SVEUČILIŠTE JOSIPA JURAJA STROSSMAYERA
U OSIJEKU
GRAĐEVINSKI FAKULTET OSIJEK

ZAVRŠNI RAD

Osijek, 15.09.2015.

Nedeljko Simonović

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TEMA: USPOREDBA REZULTATA PRORAČUNA STATIČKI NEODREĐENIH SUSTAVA DOBIVENIH RAZLIČITIM METODAMA

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JOSIP JURAJ STROSSMAYER
UNIVERSITY OF OSIJEK
FACULTY OF CIVIL ENGINEERING

FINAL PAPERWORK

SUBJECT: RESULTS COMPARISON ANALYSIS OF STATICALLY UNDERTERMINED SYSTEMS GAINED WITH DIFFERENT METHODS

Osijek, 15.09.2015.

Nedeljko Simonović

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ZNANSTVENO PODRUČJE:	TEHNIČKE ZNANOSTI
ZNANSTVENO POLJE:	TEMELJNE TEHNIČKE ZNANOSTI
ZNANSTVENA GRANA:	TEHNIČKA MEHANIKA
TEMA:	USPOREDBA REZULTATA PRORAČUNA STATIČKI NEODREĐENIH SUSTAVA DOBIVENIH RAZLIČITIM METODAMA
PRISTUPNIK:	SIMONOVIĆ NEDELJKO
NAZIV STUDIJA:	PREDDIPLOMSKI SVEUČILIŠNI STUDIJ

ZAVRŠNI RAD

Sveučilišni preddiplomski studij

Pristupnik treba usporediti rezultate proračuna dobivene: metodom sila; metodom pomaka; iteracionom metodom i numeričkim modelom na dva primjera.

Treba usporediti nekoliko veličina.

Statički sustavi su zadani u prilogu na slici. Odabrati prozivoljno presjeke i materijal nosača.

Rad treba sadržavati tekstualni dio, grafičke priloge, te popis literature i internet stranica sa koji su prikupljeni podatci za rad.

Rad treba predati u 3 primjerka (original + 2 kopije), spiralno uvezana na A4 formatu i cjelovitu elektroničku datoteku na CD-u.

Osijek, 15.09.2015.

Mentor/ica:	Predsjednik/ica Odbora za Završne i diplomske ispite:
_____	_____

SADRŽAJ: str.

UVOD O METODAMA PRORAČUNA

METODA SILA

Metoda sila se računa u 7 koraka.
1.) Potrebno je odrediti stupanj statičke neodređenosti konstrukcije (n).
2.) Statička neodređenost konstrukcije n odgovara broju veza koje je potrebno ukinuti i time transformirati statički neodređeni sustav u statički određeni sustav. Takav sustav se naziva osnovni sistem.
3.) Na mjestu oslobođenih veza j, postavljaju se nepoznate sile Xj(sile u prekobrojnim vezama) koje odgovaraju reakcijama ukinutih ležaja.
4.) Primjena danog opterećenja ili prisilnih pomaka na osnovni sistem-u smislu crtanja dijagrama unutarnjih sila. Računaju se pomaci zbog zadanog opterećenja na mjestima ukinutih veza u osnovnom sistemu. Ovi pomaci se označavaju δ_{10}, δ_{20}..., δ_{n0}
5.) Na mjestu ukinut. pridržanja-veza j u osn.sistemu, postavljaju se jedinične sile $X_j=1$. Izračunavaju se pomaci zbog ovih jediničnih sila na mjestima ukinutih veza u osnovnom sistemu. Ovi pomaci se označavaju δ_{1j}, δ_{2j}..., δ_{nj}.
6.) Računanje sila X_1 do X_n koristeći uvjete kompatibilnosti s početnom statički neodređenom konstrukcijom. Iz osnovnog sistema se vraćamo u statički neodređeni sustav. Za to nam služe jednadžbe:

$$\delta_{10} + X_1*\delta_{11} + X_2*\delta_{12} +...+ X_n*\delta_{1n} =0$$
$$\delta_{20} + X_1*\delta_{21} + X_2*\delta_{22} +...+ X_n*\delta_{2n} =0$$
$$...$$
$$\delta_{n0} + X_1*\delta_{n1} + X_2*\delta_{n2} +...+ X_n*\delta_{nn} =0$$

7.)Izračunavanje sila S na određenim mjestima na st. n. konstrukciji korištenjem slijedećih funkcijskih veza:

$$S = S_0 + X_1 \cdot S_1 + X_2 \cdot S_2 + ... + X_n \cdot S_n,$$

gdje su veličine Xj izračunate iz sistema jednadžbi danih u prethodnom koraku. S_0 je sila uslijed zadanog opterećenja ili prisilnih pomaka na osnovnom sistemu. S_j je sila uslijed jediničnih sila Xj=1 na osnovnom sistemu. Veličina S može biti moment savijanja, poprečna ili uzdužna sila, reakcija ili pomak.

Da bi izračunali površine ispod krivulja dijagrama koristimo Vereščaginov teorem:

Integral umnoška dviju neprekinutih funkcija u granicama (a,b), pri čemu je jedna funkcija linearna, jednak je umnošku površine, omeđene nelinearnom funkcijom i osi x, u granicama integracije, i ordinate linearne funkcije ispod težišta površine nelinearne funkcije.

METODA POMAKA

Metodom pomaka mogu se proračunavati i statički određeni i statički neodređeni sistemi. U metodi pomaka sve se svodi na posmatranje elemenata i čvorova.
Metodu pomaka računamo tako da odredimo pomake čvorova konstrukcije, translaciju ili rotaciju štapova. Nepoznanice su kut zaokreta čvora(φ) i translacijski pomak (u). Broj neopoznanica odgovara stupnju statičke neodređenosti.
Određuje se kutevi zaokreta štapova (Ψ) i krutost štapova (k).
Sljedeći korak je rastaviti konstrukciju na zasebne dijelove i promatrati utjecaj vanjskog opterećenja na svaki štap zasebno, a to se radi izračunom momenta upetosti.
Nakon momenta upetosti ispisuje se jednadžbe momenata na krajevima štapova. U slučaju da je obostrano upet jednadžba glasi:

$M_{xy}=k_{xy}*(4\varphi_x+2\varphi_y-6\Psi_{xy}*u)+M_{xy}'$

Kada je to štap upet samo s jedne strane jednadžba glasi:

$M_{xy}=k_{xy}*(3\varphi_x-3\Psi_{xy}*u)+M_{xy}'$

Potrebno je tako dobivene vrijednosti uvrstiti u jednadžbu ravnoteže čvora i jednadžbu rada.
-jednadžba ravnoteže čvora M_B

$\sum M_B=0$

-jednadžba rada

$\sum M_{ik}*\Psi_{ik} + P*\delta=0$

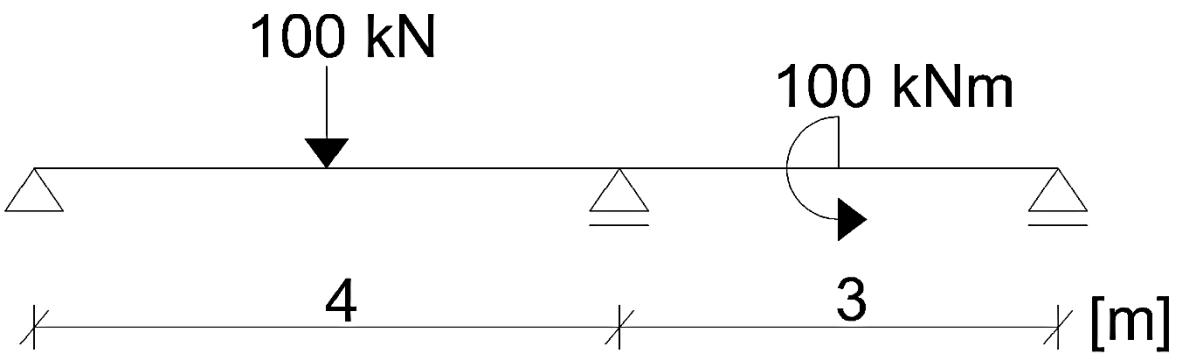
Iz jednadžbi se izračunaju nepoznanice i te vrijednosti se vraćaju u jednadžbe momenata na krajevima štapova. Te se zatim crta momenti dijagram.

CROSS- METODA

Drugim nazivom „Metoda distribucije momenta".
Postupak se provodi na grafičkoj shemi konstrukcije, nacrtamo konstrukciju, na mjestu nepoznatog kuta zaokreta ucrtamo krug ili kvadrat s razdjelim koeficijentima.
Na krajeve greda i stupova upisujemo pripadne momente upetosti, a potom,redom u proračunu, raspodijeljene i prenesene momente.
Izračunamo rezidualne momente slobodnih čvorova
Iteracije-"otpustimo" uklještenje u čvoru sa najvećim rezidualnim momentom, on se zaokreće i zauzima ravnotežni položaj, tada se neuravnotežni moment uravnoteži u priključenim štapovima u omjerima krutosti pojedinih štapova.
Pri tom uravnoteženju šaljemo dio momenta na druge krajeve priključenih štapova. Redom nastavljamo uravnoteženje na drugim slobodnim čvorovima i ponavljamo iteracije. Postupak iteracije teče tako dugo dok je

neuravnoteženi moment u svakom čvoru manji od unaprijed odabrane vrijednosti $\varepsilon \leq \Delta Mij$;
Konačni momenti na kraju štapa dobiju se zbrajanjem momenta upetosti i prirasta tijekom iteracije
Sile na krajevima štapova Tij i Nij određuju se na isti način kao kod metode pomaka.

1. PRIMJER



Dimenzije elementa: b/h=30/35[cm]
Modul elastičnosti: $3 * 10^7$ [kN/m²]

METODA SILA

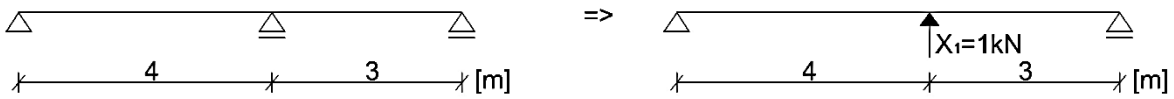
1.) Statička neodređenost
S= 2*Č-(Š+K+L) = 2*3 – (2+1+4) = -1

⇒sustav je jedanput statički neodređen
⇒metodom sila oslobađamo ležaj u sredini, postavljanjem odgovarajuće zamjenske reakcije u obliku sile X_1=1kN

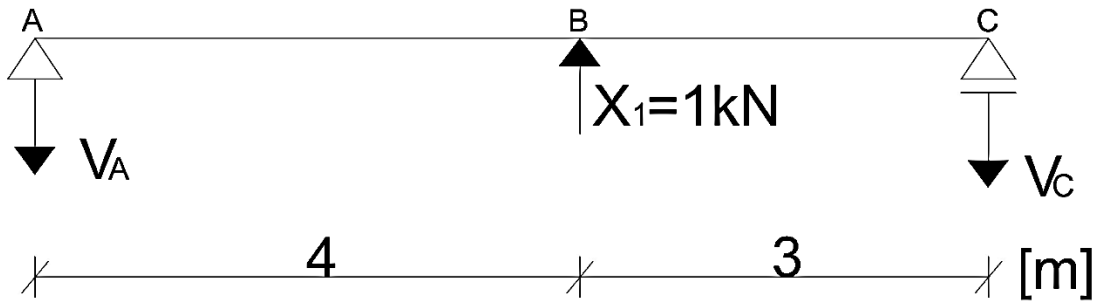
2.) Geometrijske i materijalne karakteristike
EI=E_0I_0= 3*10⁷x $\frac{0.3*0.35^3}{12}$= 32 156.25 kNm²
EA= 3*10⁷*(0.3*0.35)= 3 150 000 kN

-koeficijenti
⇒ M= $\frac{E_0I_0}{EI}$ = 1 ⇒ N=$\frac{E_0I_0}{EA}$ = 0.01

3.) Osnovni sustav



4.) Momentni dijagram za X_1=1kN



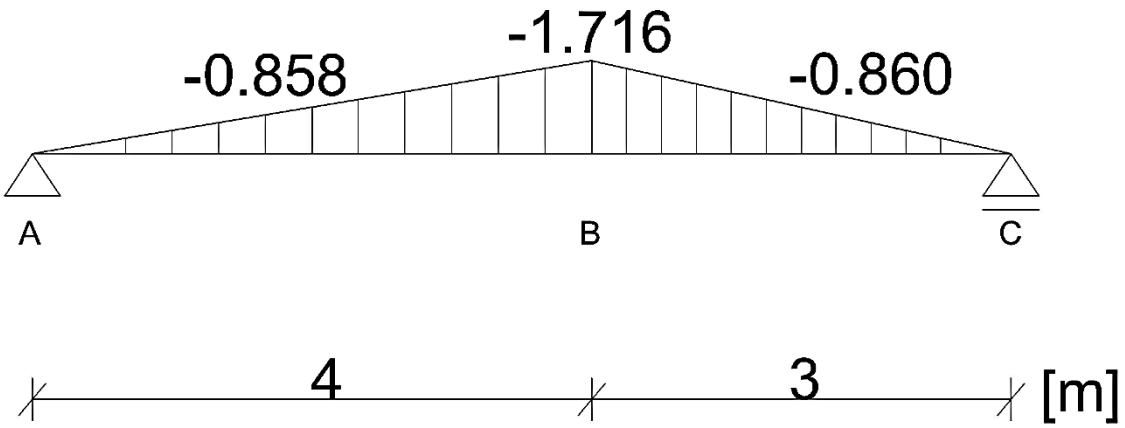
-reakcije

$\Sigma M_A=0$	$\Sigma M_C=0$	$\Sigma F_y=0$
$-V_C*7+X_1*4=0$	$V_A*7-X_1*3=0$	$1-V_A-V_C=0$
V_C=0.571kN	V_A=0.429kN	1-0.429-0.571=0

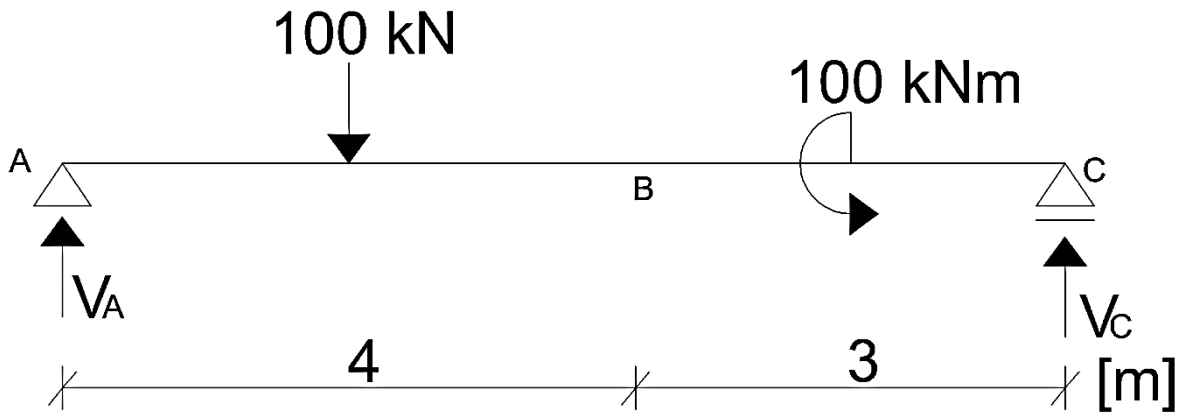
- momenti u ključnim točkama

$M_A=M_C=0$ kNm
$M_{AB/2}=-0.429*2=-0,858$ kNm
$M_B=-0.429*4=-1.716$kNm
$M_{BC/2}=-0.429*5.5 + 1*1.5=-0.860$ kNm

-momenti dijagram m_1



5.) Momentni dijagram za vanjsko opterećenje



-reakcije

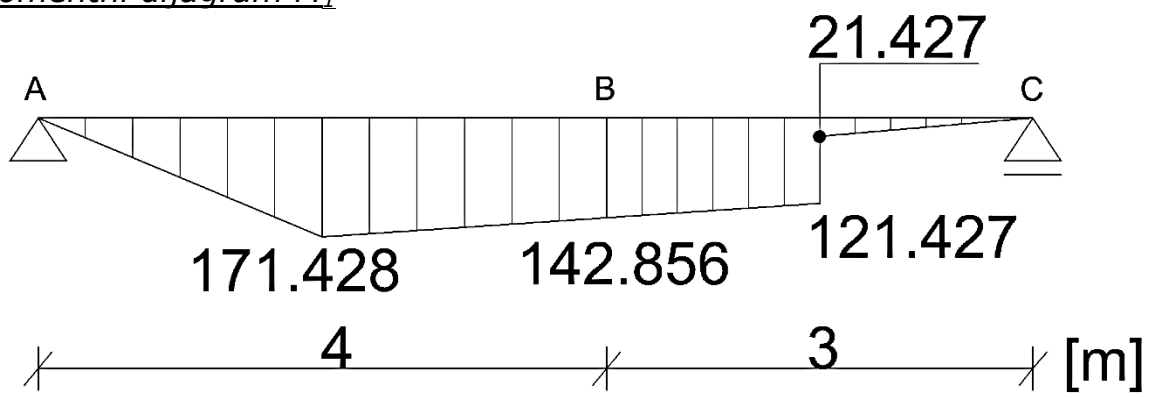
$\sum M_A=0$	$\sum M_C=0$	$\sum F_y=0$
$V_C*7+100-100*2=0$	$-V_A*7+100*5+100=0$	$V_A+V_C-100=0$
$V_C=14.286$ kN	$V_A=85.714$ kN	$85.714+14.286-100=0$
		$0=0$

- momenti u ključnim točkama

$M_A=M_C=0$ kNm
$M_{AB/2}=85.714*2=171.428$ kNm
$M_B=85.714*4-100*2=142.856$ kNm
$M^L_{BC/2}=85.714*5.5-100*3.5=121.427$ kNm

$M^{D}_{BC/2}$=121.427-100=21.427 kNm

-momentni dijagram M_1

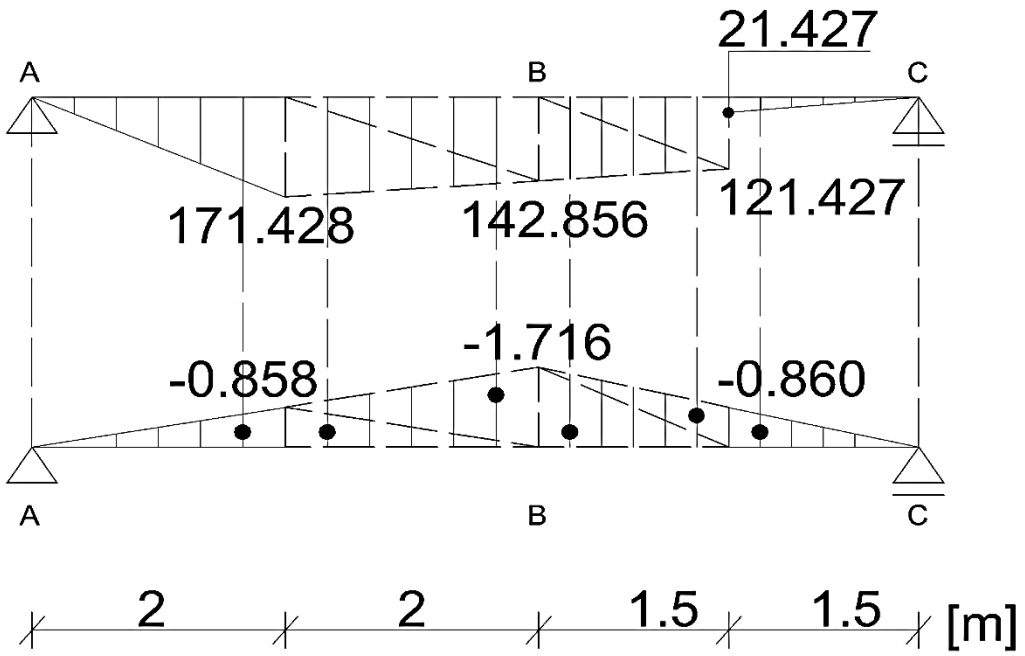


7.) Koeficijenti fleksibilnosti

$$\delta_{11}=+1*\frac{0.858*2}{2}*\frac{2}{3}0.858+1*\frac{1.716*2}{2}*\left(\frac{2}{3}1.716+\frac{1}{3}0.858\right)+1*\frac{0.858*2}{2}*\left(\frac{2}{3}0.858+\frac{1}{3}1.716\right)+1*\frac{1.716*1.5}{2}*\left(\frac{2}{3}1.716+\frac{1}{3}0.860\right)+1*\frac{0.860*1.5}{2}*\left(\frac{2}{3}0.860+\frac{1}{3}1.716\right)+$$
$$+1*\frac{0.860*1.5}{2}*\frac{2}{3}0.860$$

δ_{11}=0.491+2.454+0.982+1.841+0.739+0.370= 6.877

-kombinacija m_1 i M_1



$\delta_{10}=-1*\frac{0.858*2}{2}*\frac{2}{3}171.428-1*\frac{1.716*2}{2}*\left(\frac{2}{3}142.856+\frac{1}{3}171.428\right)-1*\frac{0.858*2}{2}*\left(\frac{2}{3}171.428+\frac{1}{3}142.856\right)-1*\frac{1.716*1.5}{2}*\left(\frac{2}{3}142.856+\frac{1}{3}121.427\right)-1*\frac{0.860*1.5}{2}*\left(\frac{2}{3}121.427+\frac{1}{3}142.856\right)-1*\frac{0.860*1.5}{2}*\frac{2}{3}21.427$

δ_{10}=-98.057-261.484-138.914-174.663-82.928-9.214= -765.26

7.) Jednadžba kontinuiteta

$\delta_{11}*X_1+\delta_{10}=0$

6.877* X_1 -765.26=0

X_1=111.278 kNm

8.) Konačni momentni dijagram

$M_A=M_C$=0 kNm

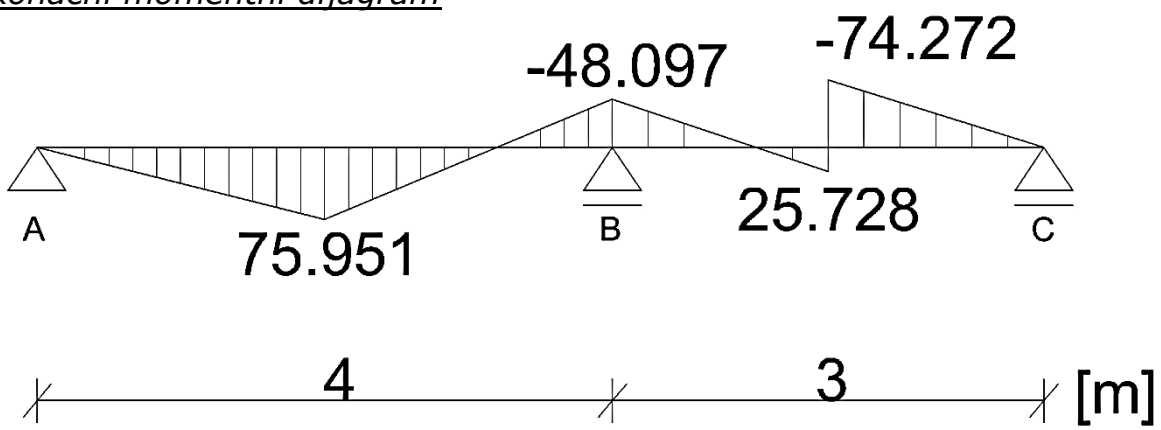
M_1=171.428-(0.858*111.278)=75.951 kNm

M_2= 142.856-(1.716*111.278)=-48.097 kNm

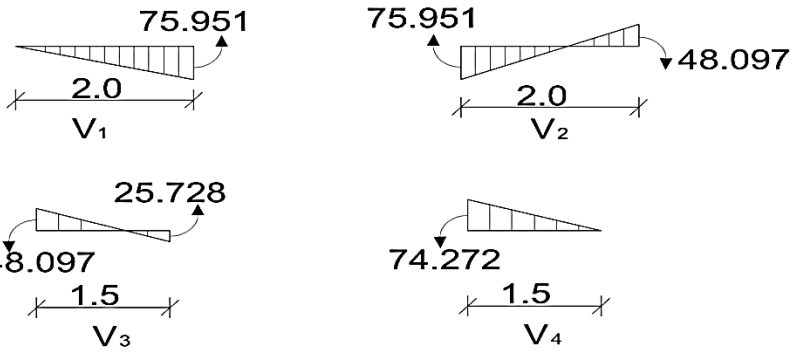
M_3^L=121.427-(0.860*111.278)=25.728 kNm

M_3^D=25.728-100=-74.272 kNm

-konačni momentni dijagram



9.) Diferencijalni M-V odnosi



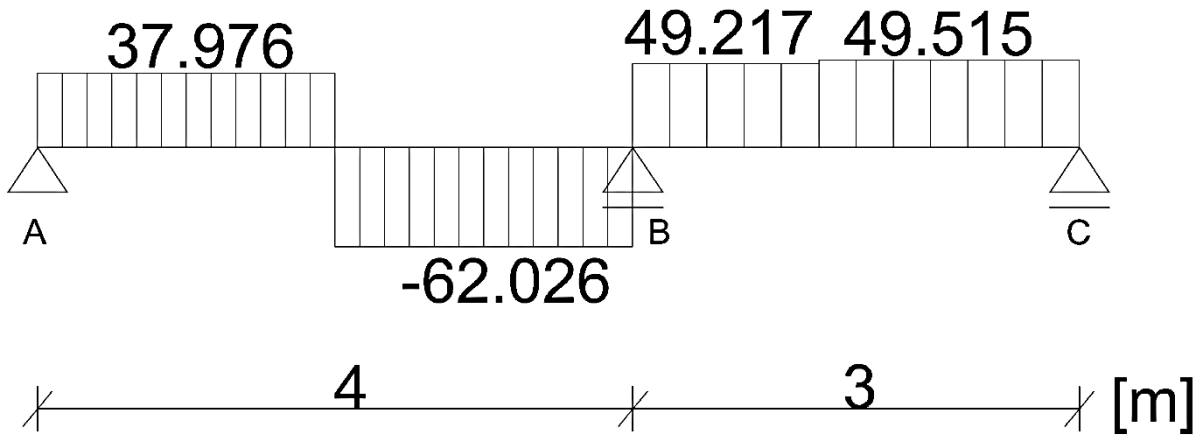
$V_1=\frac{75.951}{2}$=37.976 kN

$V_2=\frac{-75.951-48.097}{2}$=-62.026 kN

$V_1=\frac{48.097+25.728}{1.5}$=49.217 kN

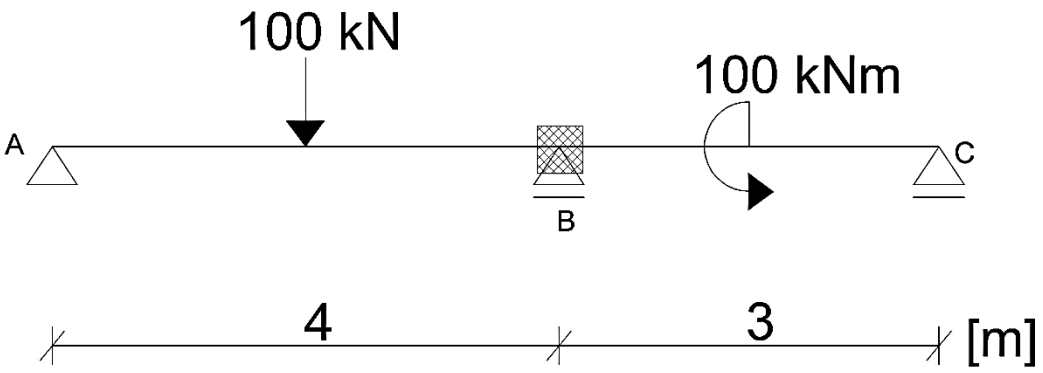
$V_1=\frac{74.272}{1.5}$= 49.515kN

-konačni V dijagram



METODA POMAKA

1.) Nepoznanice



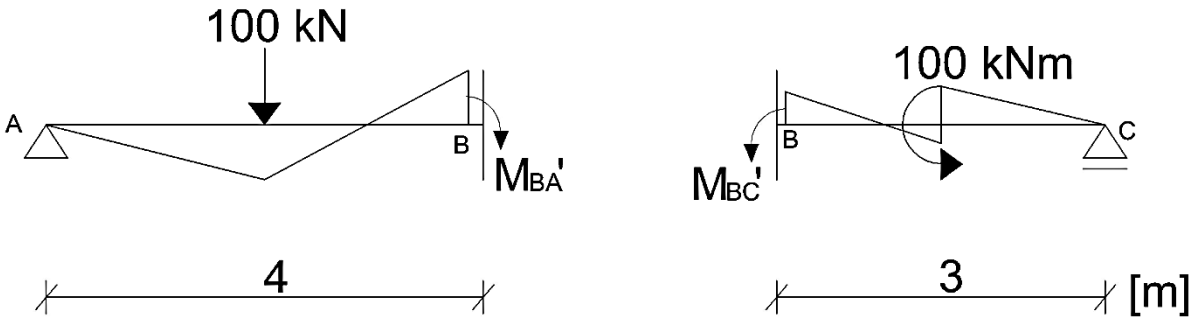
- nepoznanica φ_B

2.) Proračun krutosti elemenata

$EI_G=E_0I_0=32156.25\ kN/m^2$

$k_{AB}=\frac{1}{4}=0.25 \quad k_{BC}=\frac{1}{3}=0.333$

3.) Momenti upetosti



$M_{AB}'=M_{CB}'=0\ kNm$

$M_{BA}'=\frac{3*P*l}{16}=\frac{3*100*4}{16}=-75\ kNm$

$M_{BC}'=\frac{M}{8}=\frac{100}{8}=12.5\ kNm$

4.) Jednadžba momenata na krajevima štapova

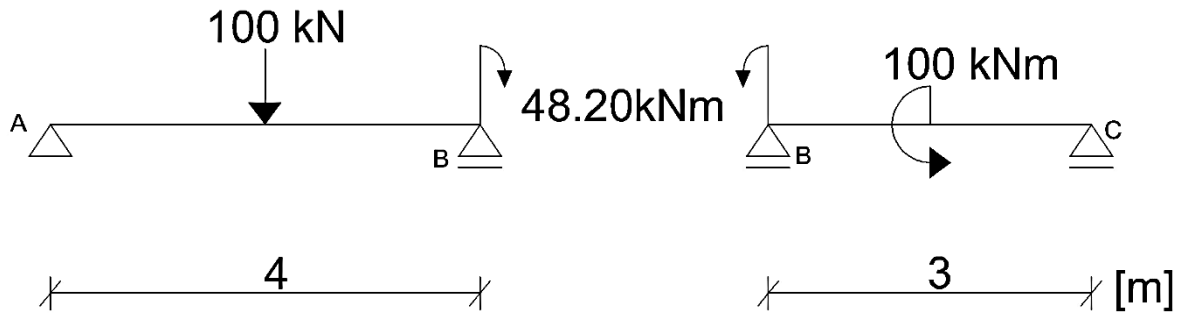
$M_{AB}=k_{AB}*(3\varphi_B-3\Psi_{AB}*u)+M_{BA}'=0.25*(3*\varphi_B-0)-75=0.75\ \varphi_B-75$
$M_{BC}=k_{BC}*(3\varphi_B-3\Psi_{BC}*u)+M_{BC}'=0.333*(3*\varphi_B-0)+12.5=\varphi_B+12.5$

5.) Jednadžba ravnoteže čvora

$\sum M_B=0$
$M_{AB}+ M_{BC}=0$
$0.75\ \varphi_B-75 + \varphi_B+12.5=0$
$\varphi_B=35.735$

- izračun momenata sa φ_B
$M_{AB}=(0.75 *35.735)-75=-48.20$ kNm
$M_{BC}=35.735+12.5=48.20$ kNm

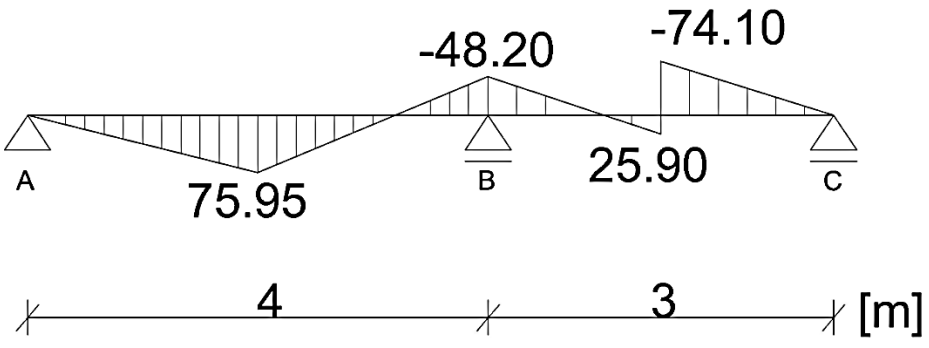
-izračun momenata na mjestima opterećenja



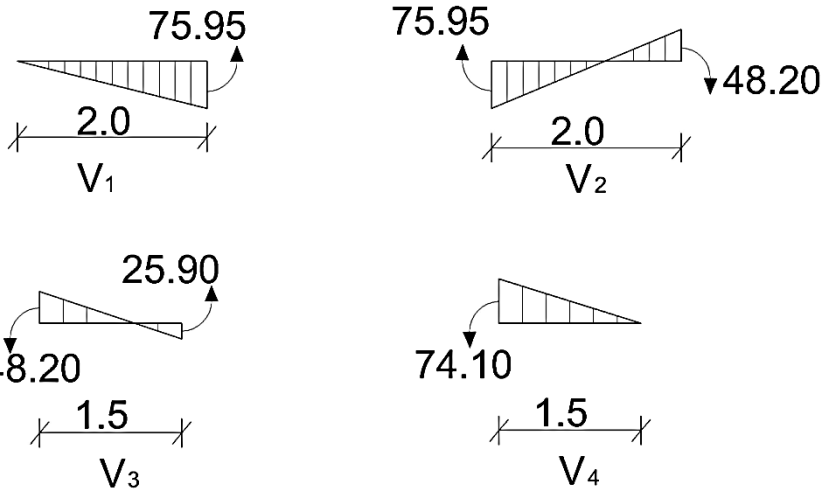
$\sum M_B=0$
$V_A*4+100*2-48.20=0$
$V_A=37.95$ kN
$M_{AB/2}=V_A*2=75.90$ kNm

$\sum M_B=0$
$V_C*3+100+48.20=0$
$V_C=49.40$ kN (-)
$M_{BC}^D=V_C*1.5=-74.1$ kNm
$M_{BC}^L=-74.1+100=25.9$ kNm

6.) Konačni M dijagram



7.) Diferencijalni M-V odnosi



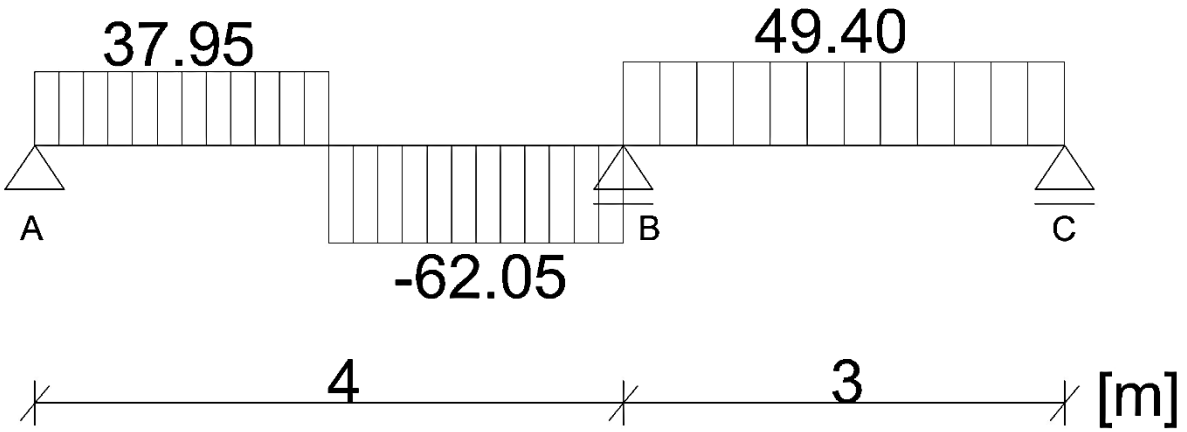
$V_1=\frac{75.95}{2}$=37.95 kN

$V_2=\frac{-75.95-48.20}{2}$=-62.05 kN

$V_1=\frac{48.20+25.90}{1.5}$=49.40 kN

$V_1=\frac{74.10}{1.5}$= 49.40kN

- konačni V dijagram



METODA CROSSA

1.) Proračun krutosti elemenata

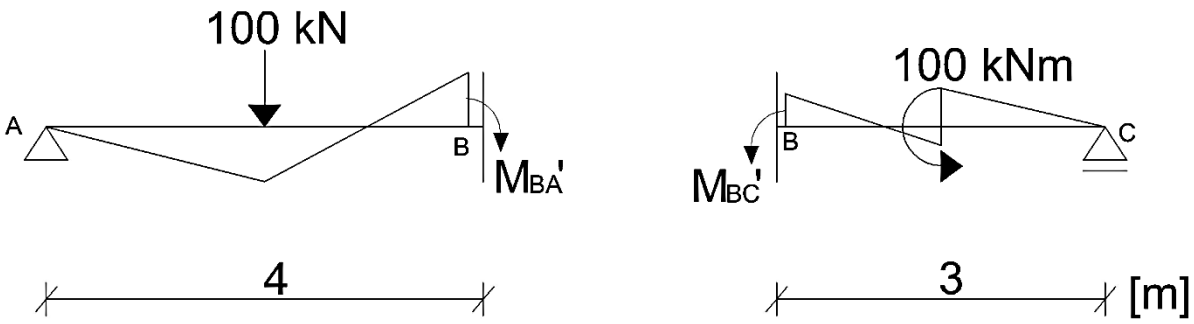
$EI_G=E_0I_0=32156.25\ kN/m^2$
$k_{AB}=\frac{1}{4}=0.25 \quad k_{BC}=\frac{1}{3}=0.333$

2.) Proračun razdjelnih koeficijenata¸

ČVOR	ŠTAP	k_i	Σk_i	μ_i	$\Sigma\mu$
B	A-B	0.25	0.538	0.429	1
	B-A	0.333		0.571	

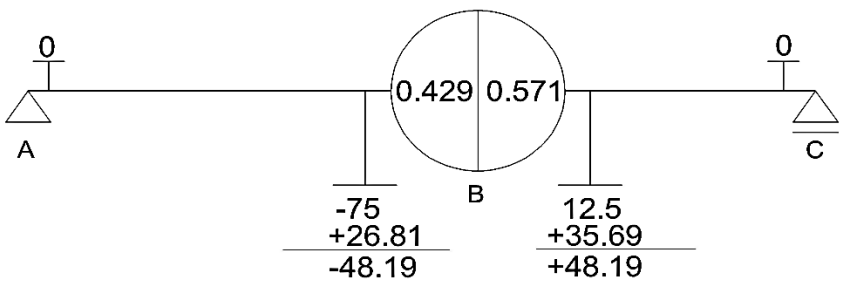
3.) Prijenosni koeficijent α=0.5

4.) Momenti upetosti

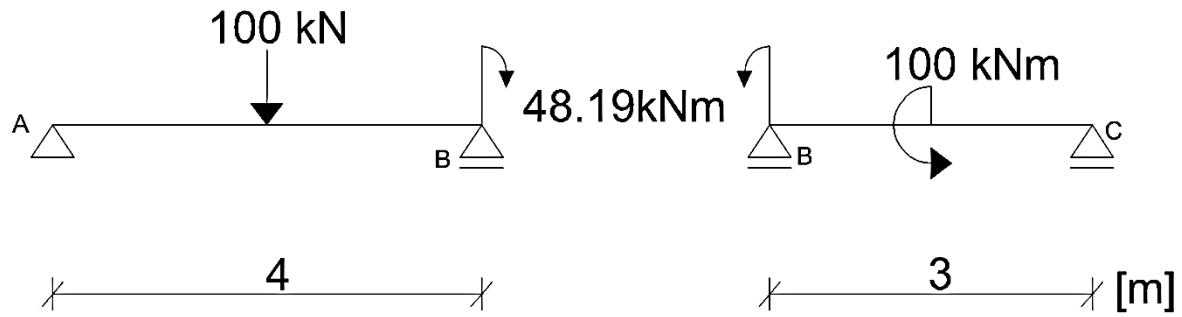


$M_{AB}'=M_{CB}'=0\ kNm$
$M_{BA}'=\frac{3*P*l}{16}=\frac{3*100*4}{16}=-75\ kNm$
$M_{BC}'=\frac{M}{8}=\frac{100}{8}=12.5\ kNm$

5.) Iteracija
$M_1=-75+12.5=-62.5\ kNm \Rightarrow$ iteracijski moment M=62.5 kNm
- postupak iteracije



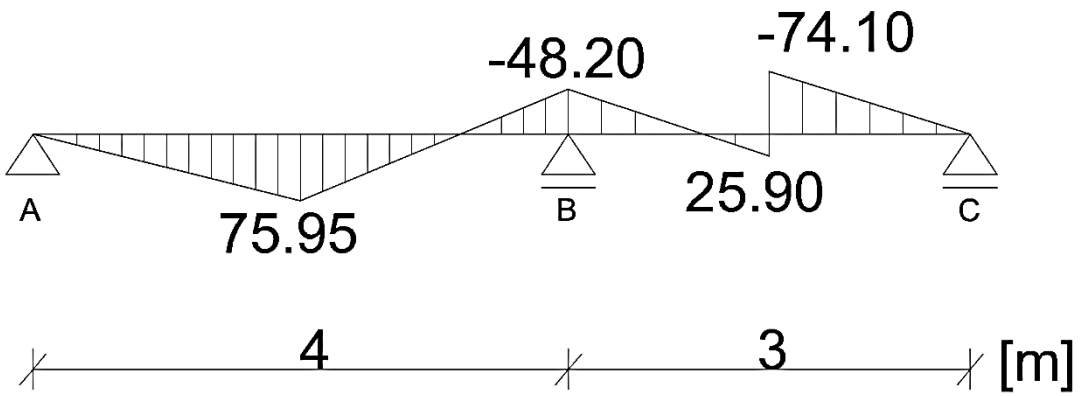
-izračun momenata na mjestima opterećenja



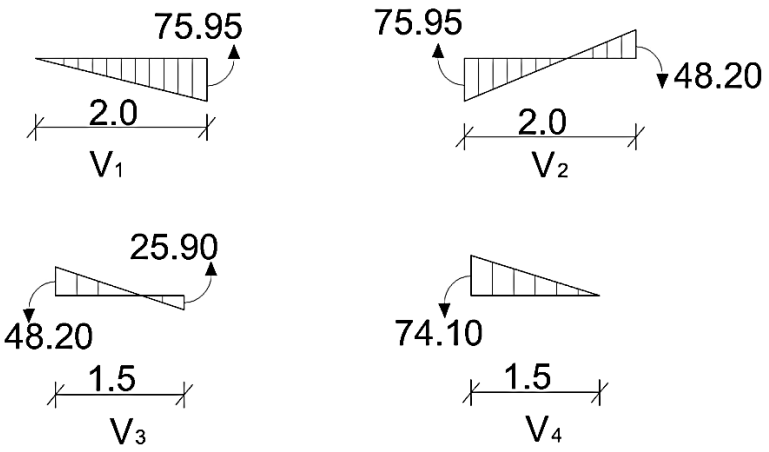
$\sum M_B=0$
$V_A*4+100*2-48.19=0$
$V_A=37.95$ kN
$M_{AB/2}=V_A*2=75.90$ kNm

$\sum M_B=0$
$V_C*3+100+48.19=0$
$V_C=49.40$ kN (-)
$M_{BC}^D=V_C*1.5=-74.1$ kNm
$M_{BC}^L=-74.1+100=25.9$ kNm

6.) Konačni M dijagram

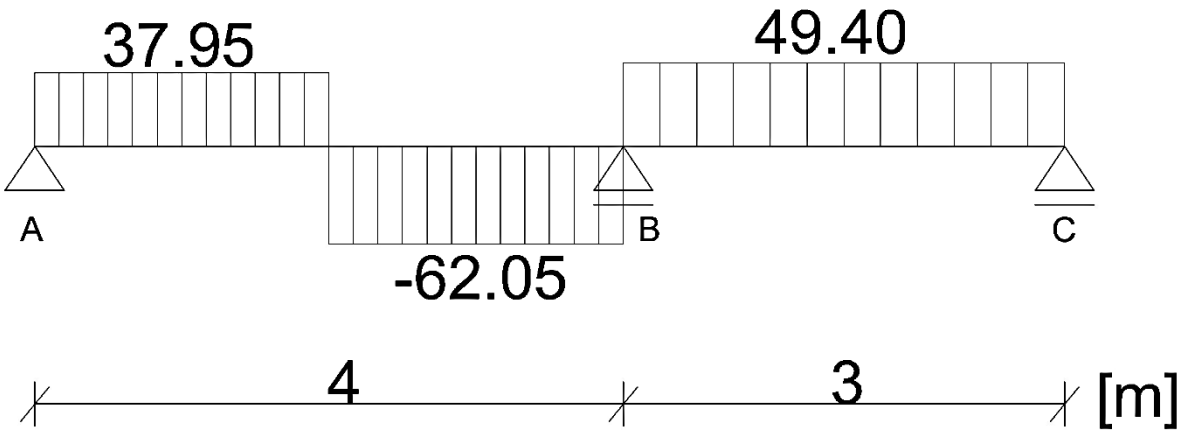


7.) Diferencijalni M-V odnosi



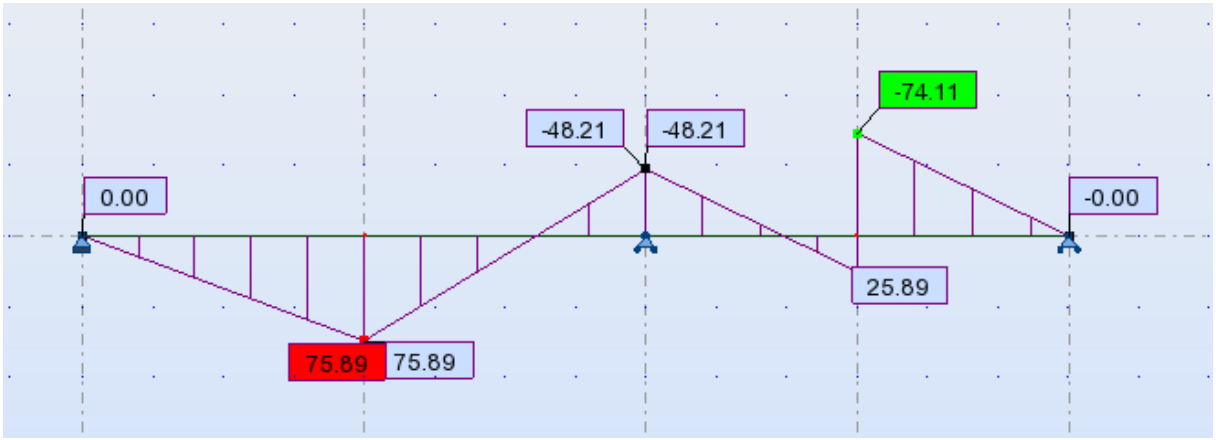
$V_1=\frac{75.95}{2}$=37.95 kN
$V_2=\frac{-75.95-48.20}{2}$=-62.05 kN
$V_1=\frac{48.20+25.90}{1.5}$=49.40 kN
$V_1=\frac{74.10}{1.5}$= 49.40kN

- konačni V dijagram

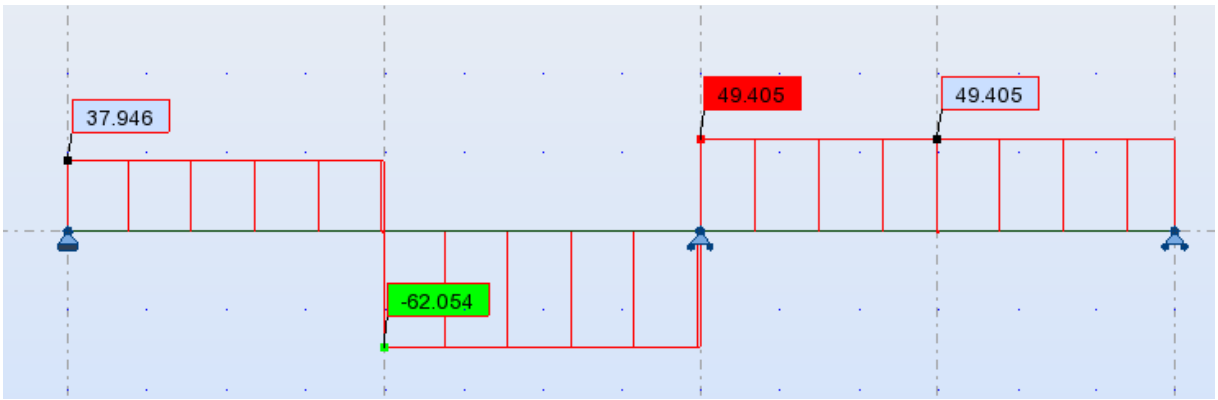


NUMERIČKI MODEL – RAČUNALNI PROGRAM „AUTODESK ROBOT STRUCTURAL ANALYSIS PROFESSIONAL 2014"

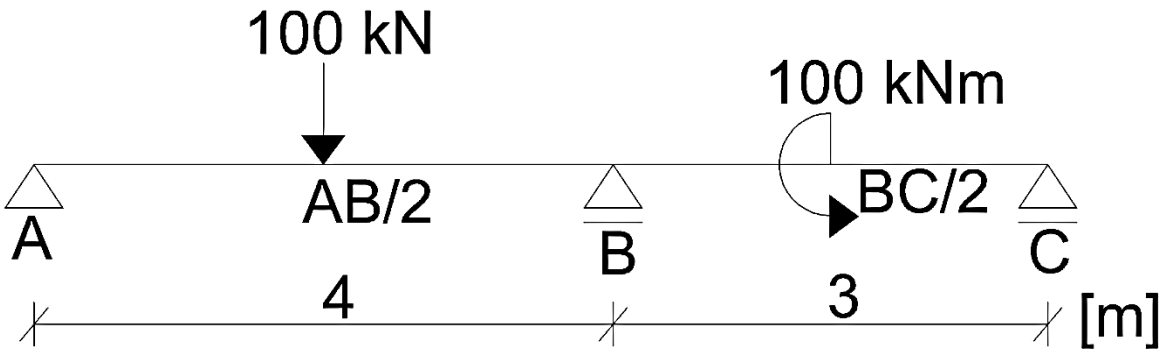
- momentni dijagram



- dijagram poprečnih sila



USPOREDBA REZULTATA PRORAČUNA STATIČKI NEODREĐENOG SUSTAVA DOBIVENIM RAZLIČITIM METODAMA

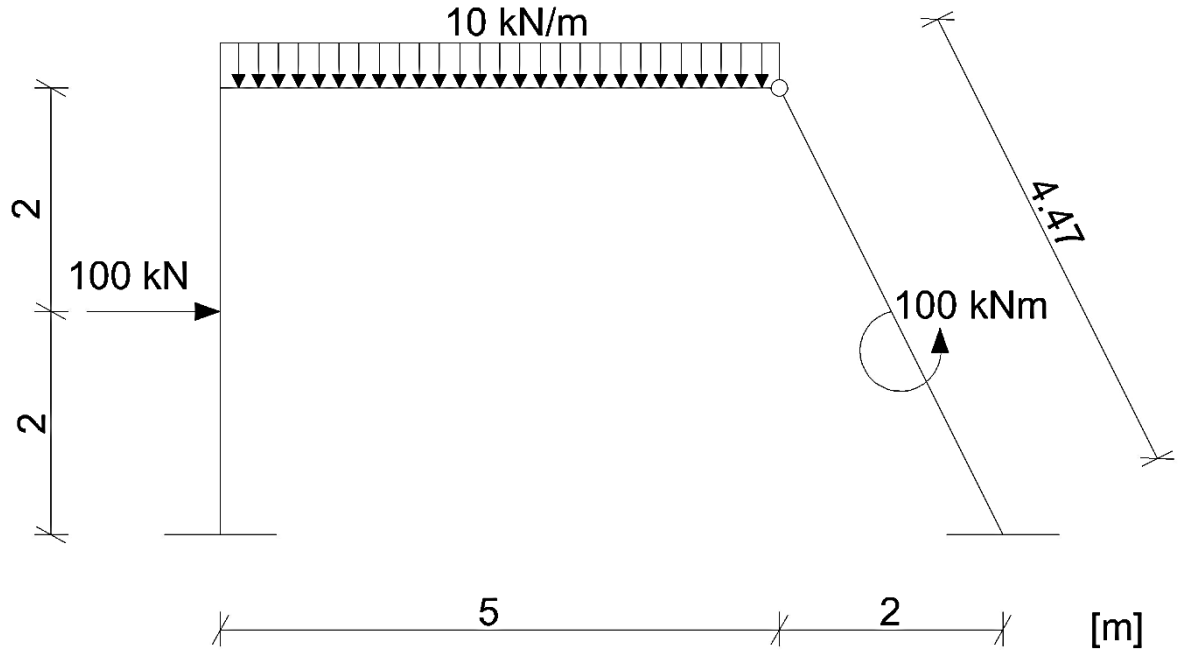


	Metoda sila	Metoda pomaka	Cross-ova metoda	ROBOT	
$M_{AB/2}$	75.91	75.95	75.95	75.89	kNm
M_B	-48.10	-48.20	-48.20	-48.21	kNm
$M_{BC/2}^{L}$	25.73	25.90	25.90	25.89	kNm
$M_{BC/2}^{D}$	-74.27	-74.10	-74.10	-74.11	kNm
R_A	37.98	37.45	37.45	37.95	kN
R_B	111.25	111.45	111.45	111.46	kN
R_C	-49.52	-49.40	-49.40	-49.41	kN

ZAKLJUČAK:

Uzimajući u obzir računalni program Robot kao referentno rješenje, može se zaključiti da „Metoda sila", „Metoda pomaka" i „Cross-ova metoda" daju odgovarajuće rezultate. Odstupanja u odnosu na rješenja iz Robota su bazirana iza decimalne točke, te samim time zanemariva. Sve tri metode su uspješno izračunale rezne sile statički neodređenog sustava.

2. PRIMJER



Dimenzije elementa:
-greda: b/h=30/30[cm]
-stupovi: b/h=30/40 [cm]
Modul elastičnosti: $3 * 10^7$ [kN/m²]

METODA SILA

1.) Statička neodređenost
S= 2*Č-(Š+K+L) = 2*4 – (3+1+6) = -2

⇒sustav je dvaput statički neodređen
⇒metodom sila oslobađamo upete ležajeve, postavljanjem odgovarajućih zamjenskih momenata u obliku momenta X_1=1kNm i X_2=2kNm

2.) Geometrijske i materijalne karakteristike

$EI_G=E_0I_0=3*10^7 x \frac{0.3*0.3^3}{12}$= 20250 kNm²

$EA_G=3*10^7*(0.3*0.3)$= 2 700 000 kN

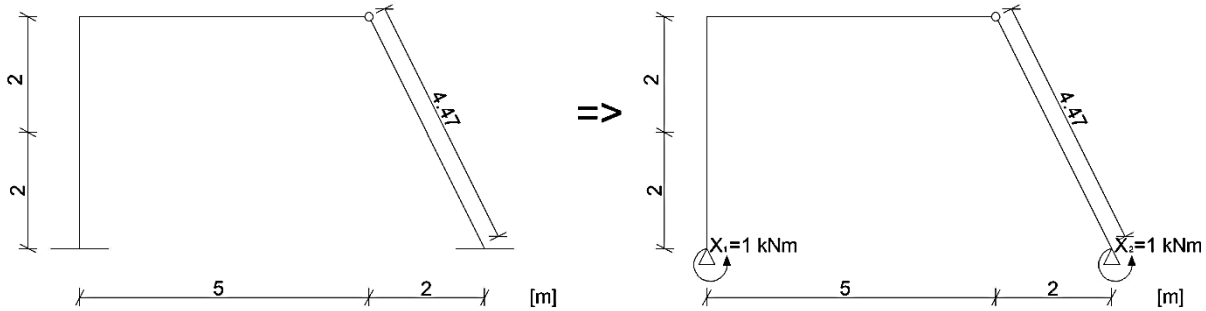
$EI_S=3*10^7 x \frac{0.3*0.4^3}{12}$= 48000 kNm²

$EA_G=3*10^7*(0.3*0.4)$= 3 600 000 kN

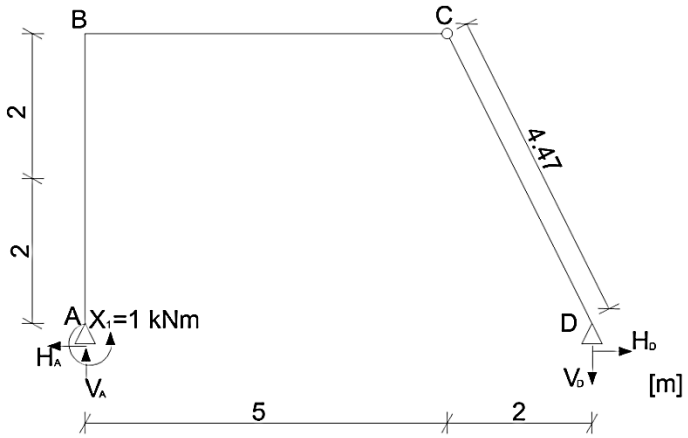
-koeficijenti

⇒ M= $\frac{E_0I_0}{EI_G}$ = 1 ; M= $\frac{E_0I_0}{EI_S}$ = 0.422 ⇒ N=$\frac{E_0I_0}{EA_G}$ = 0.008 ; N=$\frac{E_0I_0}{EA_S}$ = 0.006

3.) Osnovni sustav



4.) Momentni dijagram za X_1=1kN



-reakcije

$\Sigma M_A=0$
$-V_D*7+X_1=0$
$V_D=0.143kN$

$\Sigma M_D=0$
$-V_A*7+X_1=0$
$V_A=0.143kN$

$\Sigma M_C^{DOLJE}=0$
$H_D*4-V_D*2=0$
$H_D=0.072kN$
$H_A=H_D=0.072kN$

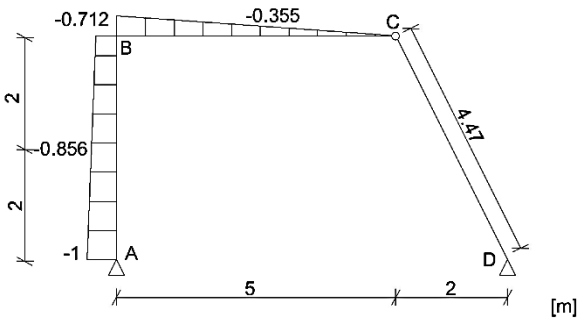
- momenti u ključnim točkama

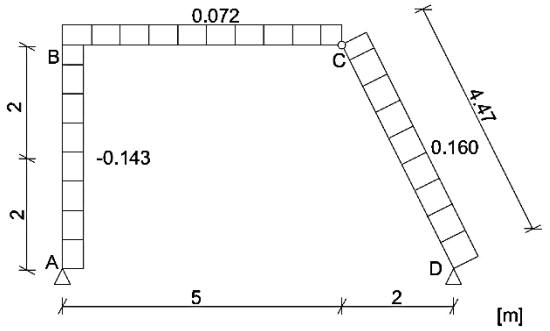
$M_D=M_C=0$ kNm
$M_A=-1$ kNm
$M_{AB/2}=-1+0.072*2=-0,856$ kNm
$M_B=-1+0.072*4=-0.712$kNm
$M_{BC/2}=-1+0.072*4+0.143*2.5=-0.355$ kNm

-uzdužne sile

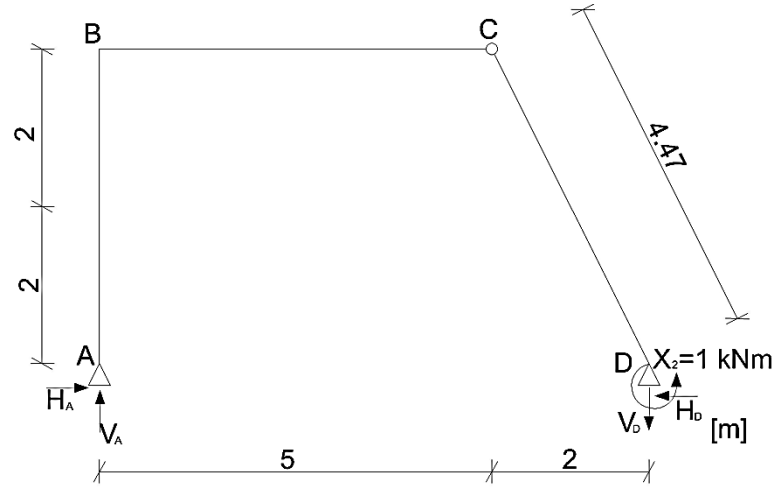
$N_{AB}=-0.143$ kN
$N_{BC}=0.072$ kN
$N_{CD}=0.143*\cos 26.57^\circ+0.072*\cos 63.43^\circ=0.160$ kN

- momentni dijagram m_1 i dijagram uzdužnih sila n_1





5.) Momentni dijagram za $X_2=1$kN



-reakcije

$\Sigma M_D=0$
$-V_A*7+X_1=0$
$V_A=0.143kN$

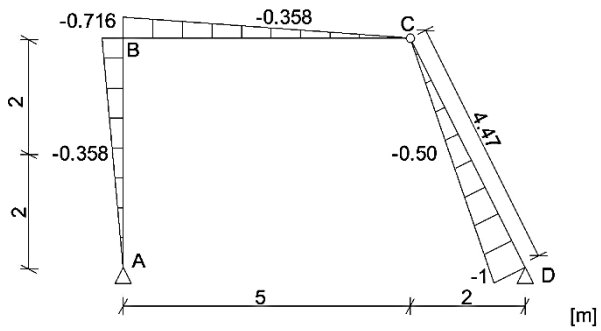
$\Sigma M_A=0$
$-V_D*7+X_1=0$
$V_D=0.143kN$

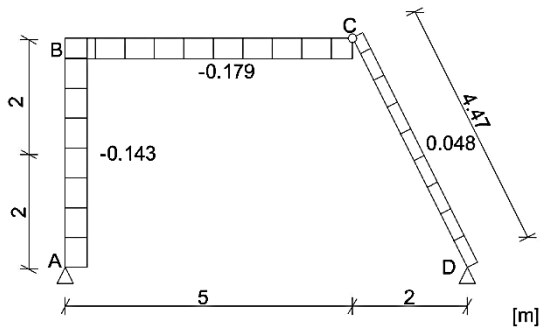
$\Sigma M_C^{DOLJE}=0$
$1-H_D*4-V_D*2=0$
$H_D=0.072kN$
$H_A=H_D=0.072kN$

- momenti u ključnim točkama
$M_A=M_C=0$ kNm
$M_{AB/2}=-0.179*2=-0.358$ kNm
$M_B=-0.179*4=-0.716$kNm
$M_{BC/2}=-0.719*4+0.143*2.5=-0.358$ kNm
$M_D=1$ kNm
$M_{DC/2}=1-0.179*2-0.143*1=0.5$ kNm

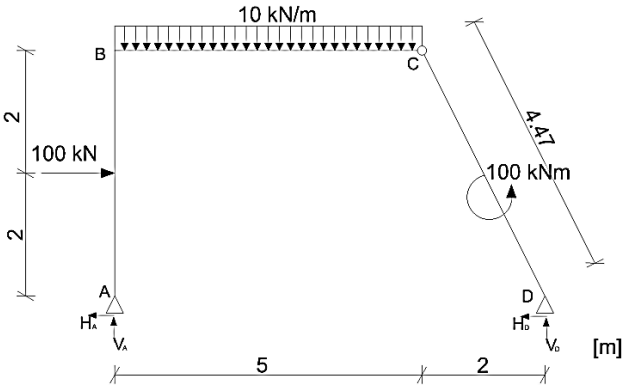
-uzdužne sile
$N_{AB}=-0.143$ kN
$N_{BC}=-0.179$ kN
$N_{CD}=0.143*\cos 26.57^\circ-0.179*\cos 63.43^\circ=0.048$ kN

- momentni dijagram m_2 i dijagram uzdužnih sila n_2





6.) Momentni dijagram za vanjsko opterećenje



-reakcije

$\sum M_D=0$
$-V_A*7-100*2+10*5*4.5+100=0$
$V_A=17.86kN$
$\sum M_C^{DOLJE}=0$
$100-H_D*4-V_D*2=0$
$H_D=41.07kN$

$\sum M_A=0$
$V_D*7+100-100*2-10*5*2.5=0$
$V_D=32.14kN$
$\sum F_X=0$
$H_A=100-41.07$
$H_A=58.93kN$

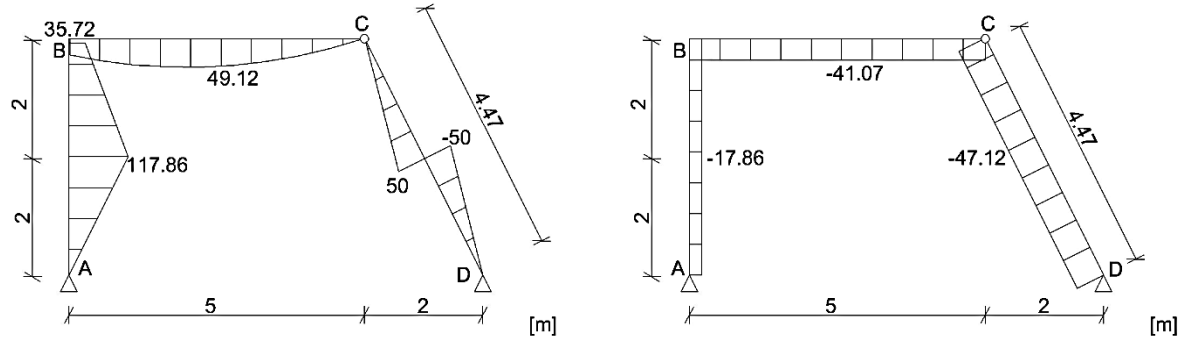
- momenti u ključnim točkama

$M_A=M_C=M_D=0$ kNm
$M_{AB/2}=58.93*2=111.86$ kNm
$M_B=58.93*4-100*2=35.72$kNm
$M_{BC/2}=58.93*4+17.86*2.5-100*2-10*2.5*1.25=49.12$ kNm
$M_{DC/2}^{DOLJE}=32.14*1-41.07*2=-50$ kNm
$M_{DC/2}^{GORE}=-50+100=50$ kNm

-uzdužne sile

$N_{AB}=-17.86$ kN
$N_{BC}=58.93-100=-47.12$ kN
$N_{CD}=-32.14*\cos 26.57^\circ-41.07*\cos 63.43^\circ=-47.12$ kN

- momentni dijagram M_1 i dijagram uzdužnih sila N_1

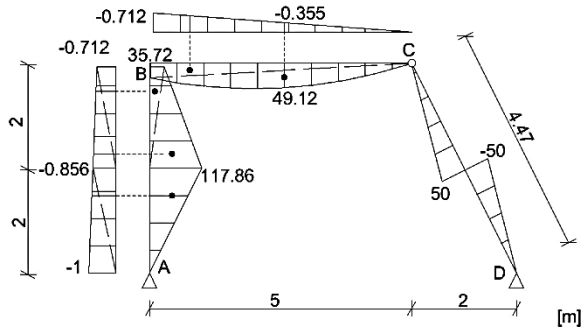


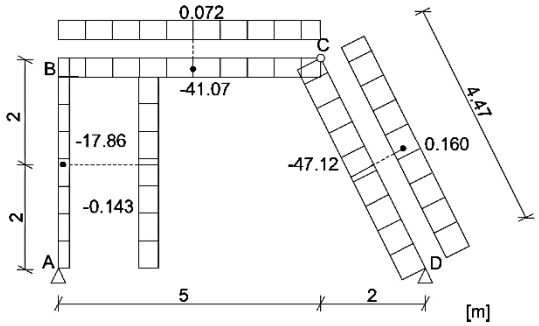
7.) Koeficijenti fleksibilnosti

$\delta_{11}=+0.422*\frac{1*4}{2}*\left(\frac{2}{3}1+\frac{1}{3}0.712\right)+0.422*\frac{0.712*4}{2}*\left(\frac{2}{3}0.712+\frac{1}{3}1\right)+1*\frac{0.712*5}{2}*\frac{2}{3}0.712+0.006*0.143*4*0.143+0.006*0.160*4.47*0.160+0.008*0.072*5*0.072$
$\delta_{11}=0.763+0.486+0.848+0.0005+0.0007+0.0002=2.098$

$\delta_{22}=+0.442*\frac{0.716*4}{2}*\frac{2}{3}0.716+1*\frac{0.716*5}{2}*\frac{2}{3}0.716+0.422*\frac{1*4.47}{2}*\frac{2}{3}1+0.006*0.143*4*0.143+0.006*0.048*4.47*0.048+0.008*0.179*5*0.179$
$\delta_{22}=0.288+0.854+0.629+0.005+0.00006+0.0013=1.773$

- koeficijent fleksibilnosti δ_{10}

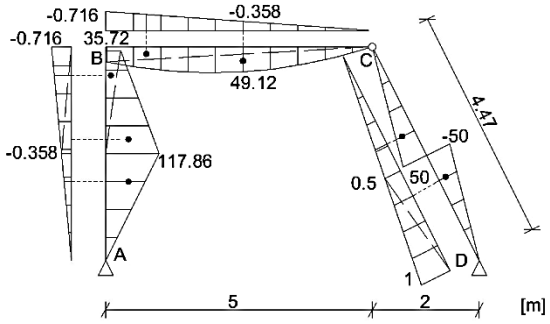


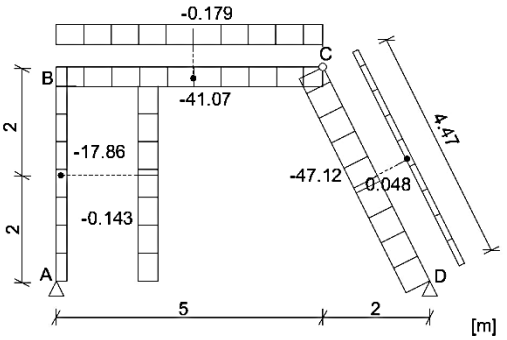


$$\delta_{10}=-0.422*\frac{117.86*2}{2}*\left(\frac{2}{3}0.856+\frac{1}{3}1\right)-0.422*\frac{117.86*2}{2}*\left(\frac{2}{3}0.856+\frac{1}{3}0.712\right)-0.422*\frac{35.72*2}{2}*\left(\frac{2}{3}0.712+\frac{1}{3}0.856\right)-1*\frac{35.72*5}{2}*\frac{2}{3}0.712-1*\frac{2}{3}*31.25*5*0.355+0.006*0.143*4*17.86-0.006*0.160*4.47*47.12-0.008*0.072*5*41.07$$

δ_{10}=-44.962-40.187-11.456-42.388-36.979+0.061-0.202-0.116=-111.231

- koeficijent fleksibilnosti δ_{20}

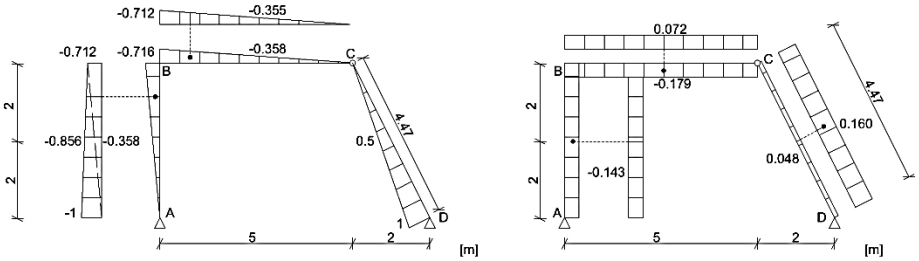




$$\delta_{20}=-0.422*\frac{117.86*2}{2}*\frac{2}{3}0.358-0.422*\frac{117.86*2}{2}*\left(\frac{2}{3}0.358+\frac{1}{3}0.716\right)-0.422*\frac{35.72*2}{2}*\left(\frac{2}{3}0.716+\frac{1}{3}0.358\right)-0.422*\frac{50*2.235}{2}*\left(\frac{2}{3}0.5+\frac{1}{3}1\right)+0.422*\frac{50*2.235}{2}*\frac{2}{3}0.5-1*\frac{35.72*5}{2}*\frac{2}{3}0.716-1*\frac{2}{3}31.25*5*0.358+0.006*0.143*17.86+0.008*0.179*5*41.07-0.006*4.47*0.048*47.12$$

δ_{20}=-11.871-23.741-8.994-15.72+7.860-42.626-37.292+0.061+0.294-0.061
=-132.09

- koeficijent fleksibilnosti δ_{12}



$\delta_{12}=\delta_{21}=+0.422*\frac{0.716*4}{2}*\left(\frac{2}{3}0.712+\frac{1}{3}1\right)+1*\frac{0.716*5}{2}*\frac{2}{3}0.712+0.006*0.143*4*0.143+0.006*0.160*0.048*4.47-0.008*0.072*5*0.179$

$\delta_{12}=\delta_{21}=0.448+0.850+0.0005+0.0002-0.0005=1.338$

8.) Jednadžba kontinuiteta

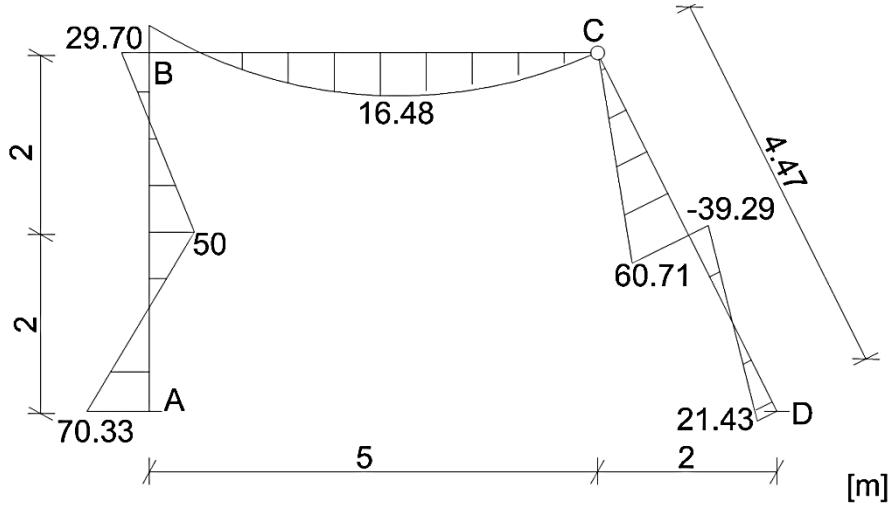
$\delta_{11}*X_1+\delta_{12}*X_2+\delta_{10}=0$
$\delta_{21}*X_1+\delta_{22}*X_2+\delta_{20}=0$
$2.098X_1+1.338X_2-176.231=0$
$1.338X_1+1.778X_2-132.09=0$
$X_1=70.33$ kNm
$X_2=21.43$ kNm

9.) Izračun momenta i uzdužnih sila

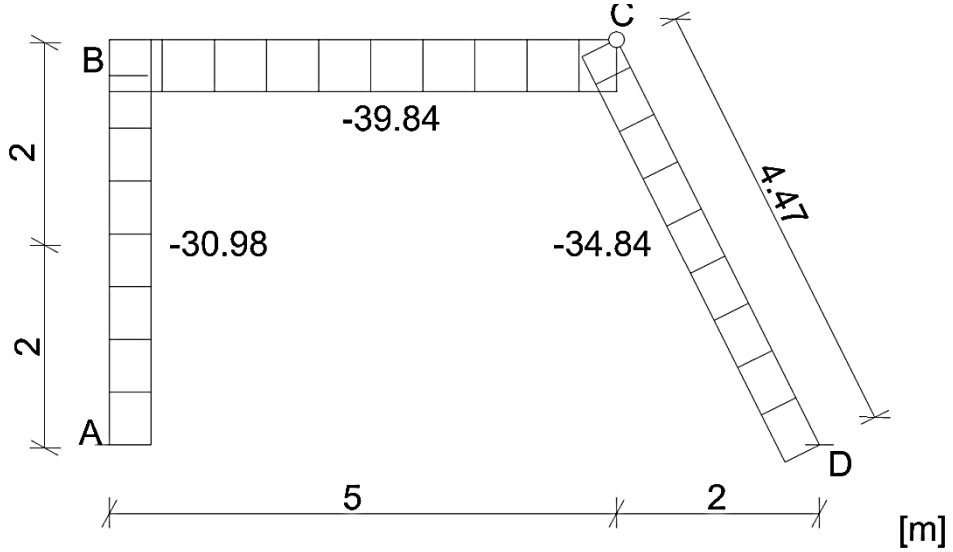
$M_C=0$ kNm
$M_A=-1*70.33=-70.33$ kNm
$M_{AB/2}=117.86-0.856*70.33-0.716*21.43=50$ kNm
$M_B=35.72-0.712*70.33-0.716*21.43=-29.70$ kNm
$M_{BC/2}=49.12-0.355*70.33-0.358*21.43=16.48$ kNm
$M_D=21.43$ kNm
$M_{CD/2}^{DOLJE}=-50+0.5*21.43=-39.29$ kNm
$M_{CD/2}^{GORE}=50+0.5*21.43=60.71$ kNm

$N_{AB}=-17.86-0.143*70.33-0.143*21.43=-30.98$ kN
$N_{BC}=-41.07+0.072*70.33-0.179*21.43=-39.84$ kN
$N_{CD}=-47.12+0.160*70.33+0.048*21.43=-34.84$ kN

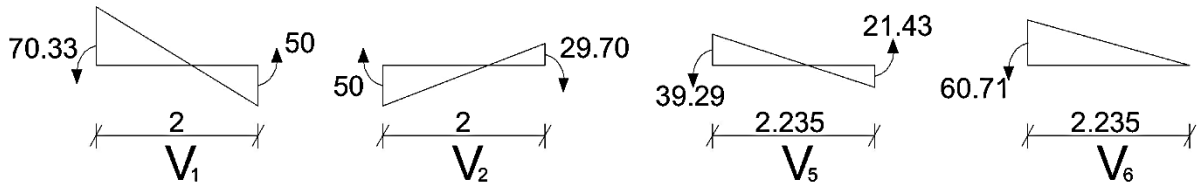
*Napomena: za konačni momentni dijagram potrebno je mjesto (x) ekstremnog momenta iz V dijagrama!

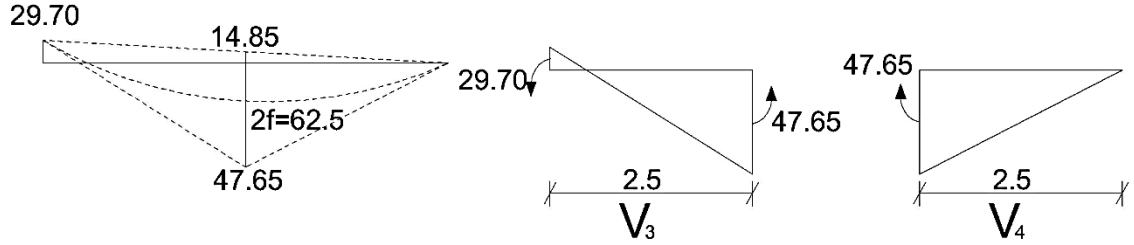


-konačni dijagram uzdužnih sila



10.) Diferencijalni M-V odnosi



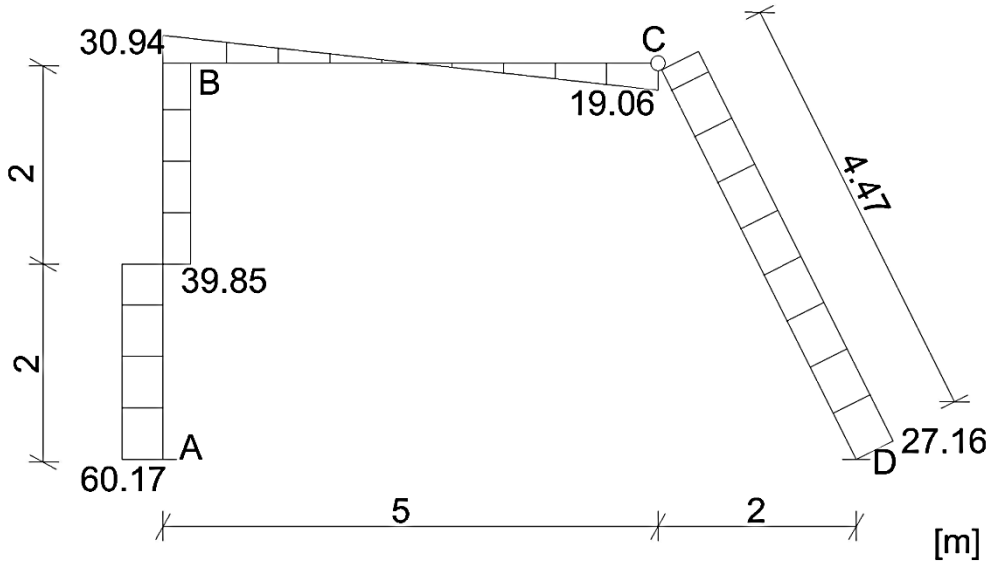


$V_1=\frac{70.33+50}{2}$=60.17 kN $V_2=\frac{-50-29.70}{2}$=-39.85 kN

$V_3=\frac{29.70+47.65}{2.5}$=30.94 kN $V_4=\frac{-47.65}{2.5}$=-19.06 kN

$V_5=\frac{39.29+21.43}{2.235}$=27.17 kN $V_6=\frac{60.71}{2.235}$=27.17 kN

-konačni dijagram poprečnih sila

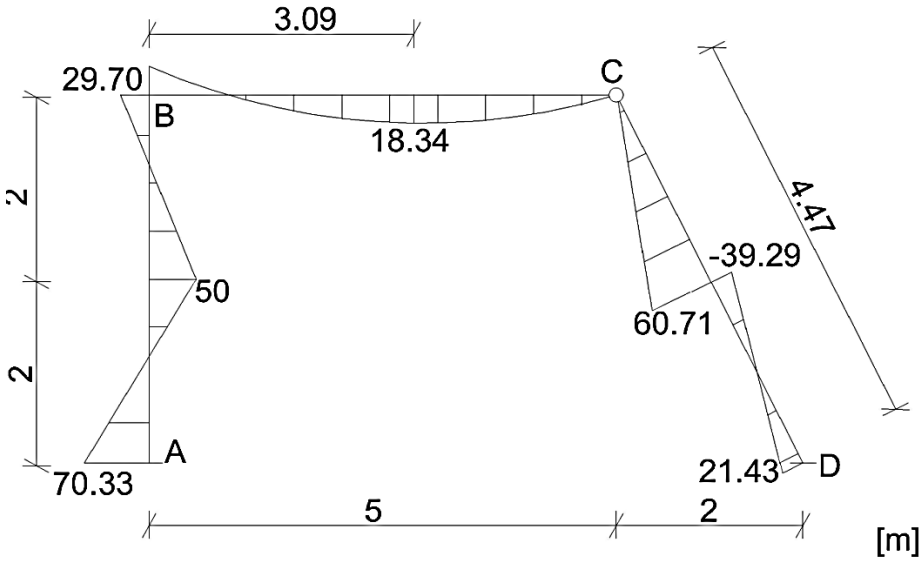


-mjesto ekstremnog momenta M_x

$\frac{50}{5}=\frac{30.94}{x}$ => $x=3.09m$

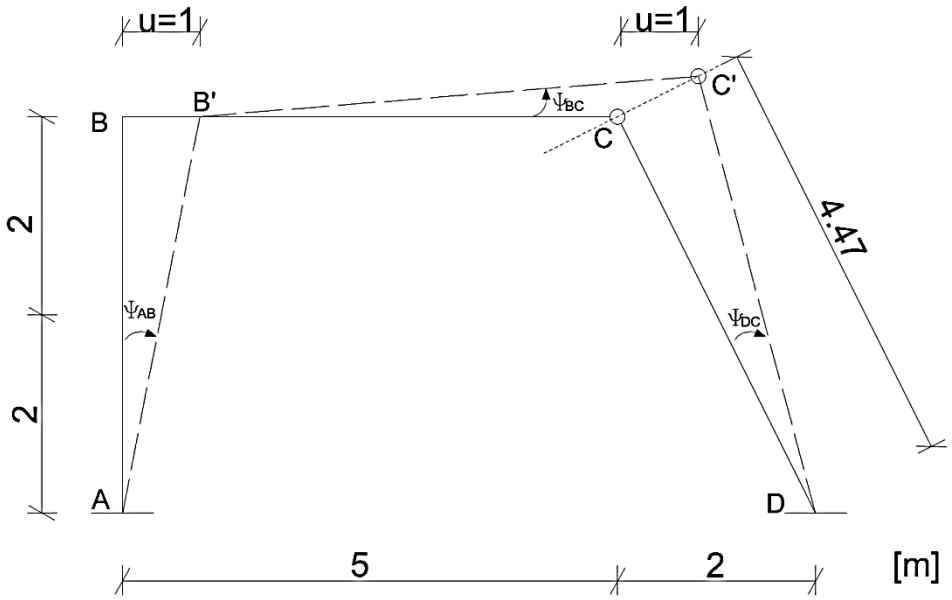
M_x=30.98*3.09+60.17*4-70.33-100*2-10*3.09*1.545=18.34 kNm

- konačni momentni dijagram



METODA POMAKA

1.) Nepoznanice



-nepoznanice φ_B i u

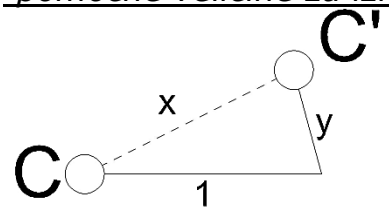
2.) Proračun krutosti elemenata i kuteva zaokreta

$EI_G=E_0I_0=3*10^7x\ \frac{0.3*0.3^3}{12}=$ 20250 kNm²

$EI_S=3*10^7x\ \frac{0.3*0.4^3}{12}=$ 48000 kNm²

$k_{AB}=\frac{48000}{20250*4}=0.593$ $k_{BC}=\frac{1}{5}=0.2$ $k_{CD}=\frac{48000}{20250*4.47}=0.53$

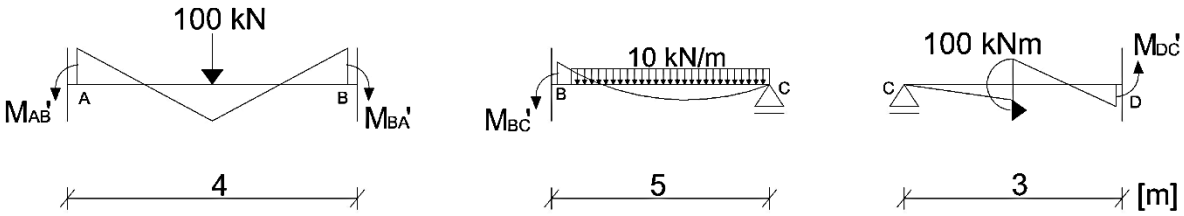
-pomoćne veličine za izračun kuteva zaokreta



$x=\frac{1}{\sin\alpha}=1.12m$ $y=\frac{1}{tg\alpha}=0.5m$

$\Psi_{AB}=\frac{-1}{4}=-0.25$ $\Psi_{BC}=\frac{0.5}{5}=0.1$ $\Psi_{DC}=\frac{-1.12}{4.47}=-0.25$

3.) Momenti upetosti



$M_{BA}'=\frac{3*P*l}{8}=\frac{3*100*4}{8}$=-50 kNm M_{AB}'=-50 kNm

$M_{BC}'=\frac{q*l^2}{8}=\frac{10*5^2}{8}$=31.25 kNm M_{CB}'=0 kNm

$M_{BC}'=\frac{M}{8}=\frac{100}{8}$=12.5 kNm M_{CD}'=0 kNm

4.) Jednadžba momenata na krajevima štapova

$M_{AB}=k_{AB}*(4\varphi_A+2\varphi_B-6\Psi_{AB}*u)+M_{AB}'=1.186\ \varphi_B+0.89u+50$
$M_{BA}=k_{AB}*(4\varphi_B+2\varphi_A-6\Psi_{AB}*u)+M_{BA}'=2.372\ \varphi_B+0.89u-50$
$M_{BC}=k_{BC}*(3\varphi_B-3\Psi_{BC}*u)+M_{BC}'=0.6\varphi_B-0.06u+31.25$
$M_{DC}=k_{DC}*(3\varphi_D-3\Psi_{DC}*u)+M_{DC}'=0.398u+12.50$

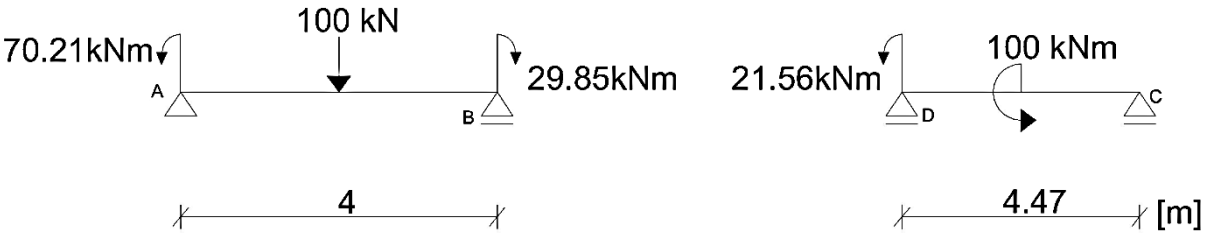
5.) Jednadžba ravnoteže čvora i jednadžba rada

-jednadžba ravnoteže čvora M_B
$\sum M_B=0$
$M_{BA}+M_{BC}=0$
$2.372\ \varphi_B+0.89u-50+0.6\varphi_B-0.06u+31.25=0$
$2.972\ \varphi_B+0.83u-18.75=0$
$\varphi_B=6.309-0.279u$ .....(1)

-jednadžba rada
$\sum M_{ik}*\Psi_{ik}+P*\delta=0$
$\Psi_{AB}*(M_{AB}+M_{BA})+\Psi_{BC}*M_{BC}+\Psi_{DC}*M_{DC}+100*0.5-100*0.25-10*5*0.25=0$
$-0.25*(3.558\varphi_B+1.78u)+0.1*(0.6\varphi_B-0.06u+31.259-0.35*(0.398u+12.5)$
$+50-25-12.5=0$
(1)..... $\varphi_B=6.309-0.279u \Rightarrow$ uvrstimo u jednadžbu
$\Rightarrow -5.236+0.232u-0.551u+12.5=0$
$u=22.771$
$\varphi_B=-0.044$

- izračun momenata sa φ_B i u
$M_{AB}=1.186\ \varphi_B+0.89u+50=70.21$ kNm
$M_{BA}=2.372\ \varphi_B+0.89u-50=-29.84$ kNm
$M_{BC}=0.6\varphi_B-0.06u+31.25=29.86$ kNm
$M_{DC}=0.398u+12.50=21.56$ kNm

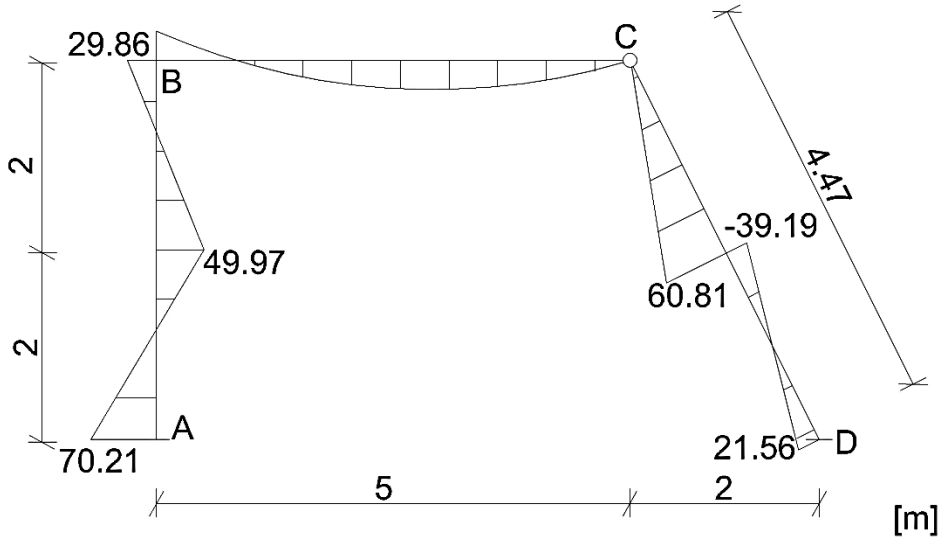
-izračun momenata na mjestima opterećenja



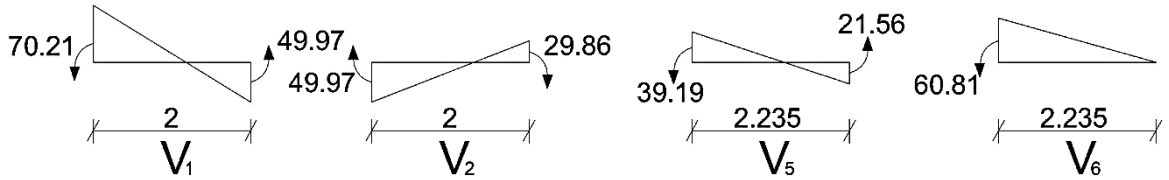
$\Sigma M_B=0$
$-V_A*4+100*2-29.85+70.21=0$
$V_A=60.09$ kN
$M_{AB/2}=V_A*2-70.21=49.97$ kNm

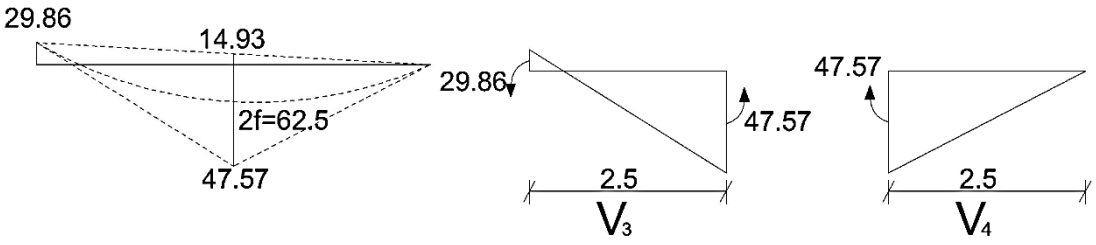
$\Sigma M_C=0$
$-V_D*4.47+100+21.56=0$
$V_D=27.19$ kN
$M_{BC}^{L}=V_D*2.235-21.56=39.19$ kNm
$M_{BC}^{L}=39.19-100=-60.81$ kNm

- momentni dijagram bez ekstremnog momenta



6.) Diferencijalni M-V odnosi



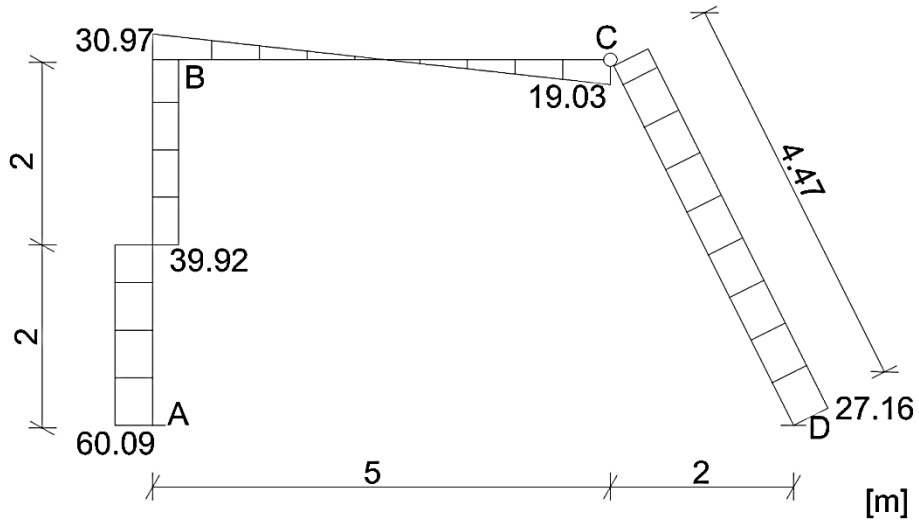


$V_1=\frac{70.21+49.97}{2}$=60.09 kN $V_2=\frac{-49.97-29.86}{2}$=-39.92 kN

$V_3=\frac{29.86+47.57}{2.5}$=30.97 kN $V_4=\frac{-47.57}{2.5}$=-19.03 kN

$V_5=\frac{39.19+21.56}{2.235}$=27.18 kN $V_6=\frac{60.81}{2.235}$=27.21 kN

-konačni dijagram poprečnih sila

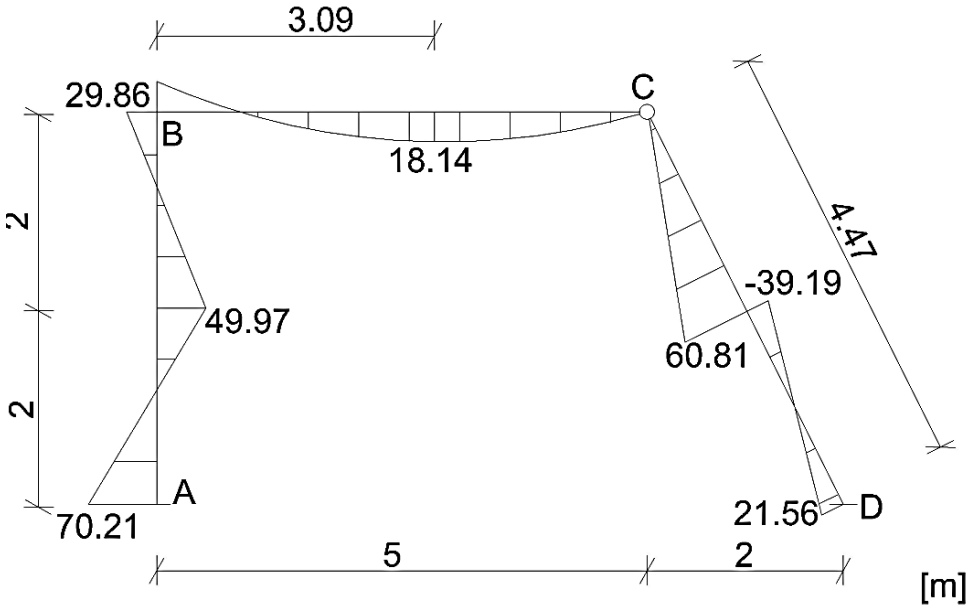


-mjesto ekstremnog momenta M_x

$\frac{50}{5}=\frac{30.97}{x}$ => $x=3.09m$

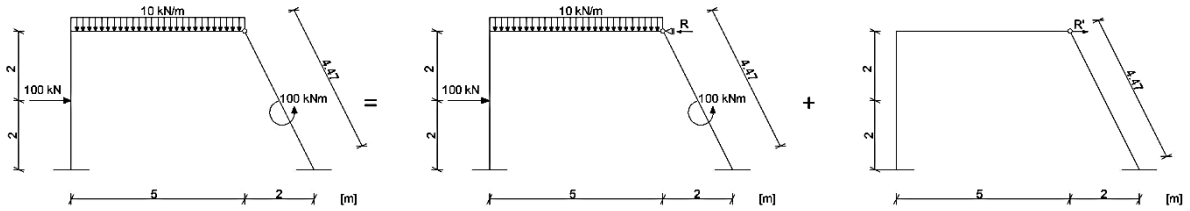
M_x=30.98*3.09+60.09*4-70.21-100*2-10*3.09*1.545=18.14 kNm

- konačni momentni dijagram



CROSS-OVA METODA

-postupak Cross-ove metode na translatorno pomičnom sustavu



1.) Proračun krutosti elemenata i kuteva zaokreta

$EI_G = E_0 I_0 = 3*10^7 x \frac{0.3*0.3^3}{12} = 20250\ kNm^2$

$EI_S = 3*10^7 x \frac{0.3*0.4^3}{12} = 48000\ kNm^2$

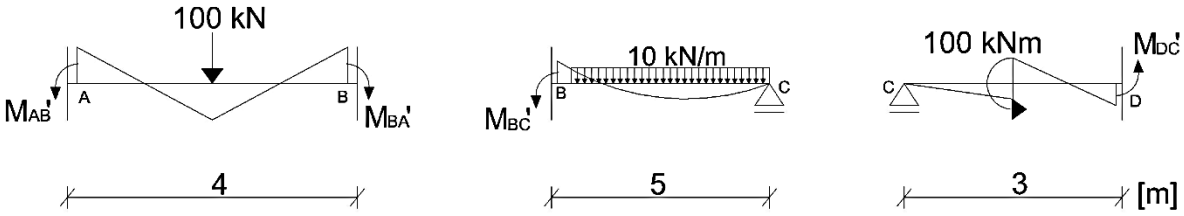
$k_{AB} = \frac{48000}{20250*4} = 0.593 \quad k_{BC} = \frac{1}{5} = 0.2 \quad k_{BC}' = 0.2*0.75 = 0.15$

2.) Razdjelni koeficijenti

ČVOR	ŠTAP	k_i	Σk_i	μ_i	$\Sigma \mu_i$
B	B-A	0.593	0.743	0.798	1
	B-C	0.15		0.202	

3.) Prijenosni koeficijent $\alpha = 0.5$

4.) Momenti upetosti



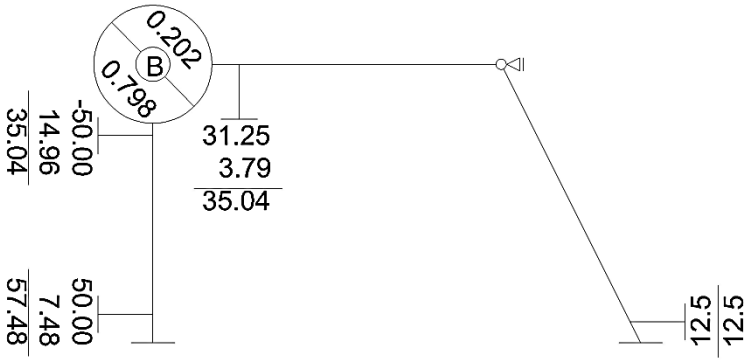
$M_{BA}' = \frac{3*P*l}{8} = \frac{3*100*4}{8} = -50\ kNm \qquad M_{AB}' = -50\ kNm$

$M_{BC}' = \frac{q*l^2}{8} = \frac{10*5^2}{8} = 31.25\ kNm \qquad M_{CB}' = 0\ kNm$

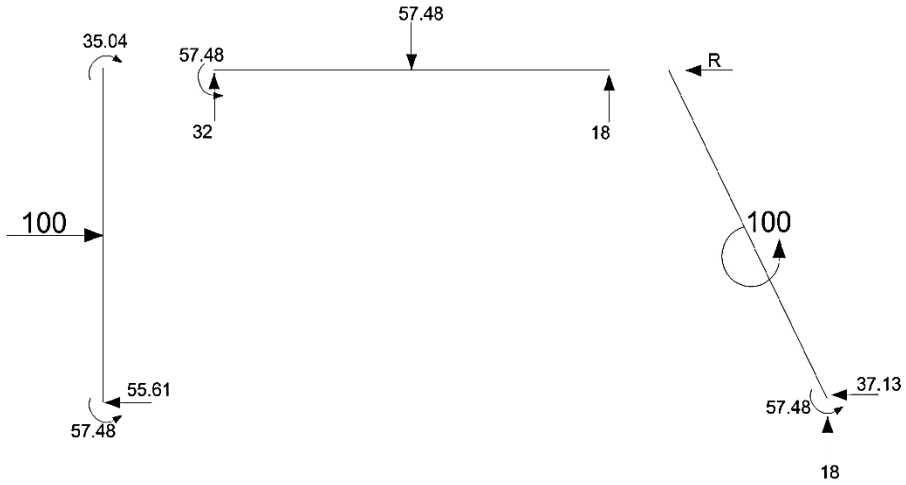
$M_{BC}' = \frac{M}{8} = \frac{100}{8} = 12.5\ kNm \qquad M_{CD}' = 0\ kNm$

5.) Iteracija na nepomičnom sustavu

$M_1 = -50 + 31.25 = -18.75\ kNm \Rightarrow$ iteracijski moment $M = 18.75\ kNm$



- određivanje sile R



$\sum F_X=0$
R=100-55.61-37.13=7.26 kN

6.) Iteracija na pomičnom sustavu

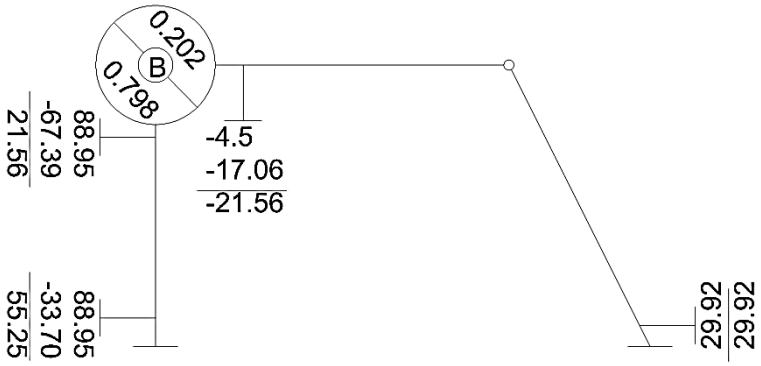
- kut zaokreta štapova iz „Metode Pomaka" za u=100

$\Psi_{AB}=\frac{-100}{4}$=-25 $\Psi_{BC}=\frac{50}{5}$=10 $\Psi_{DC}=\frac{-112}{4.47}$=-25.06

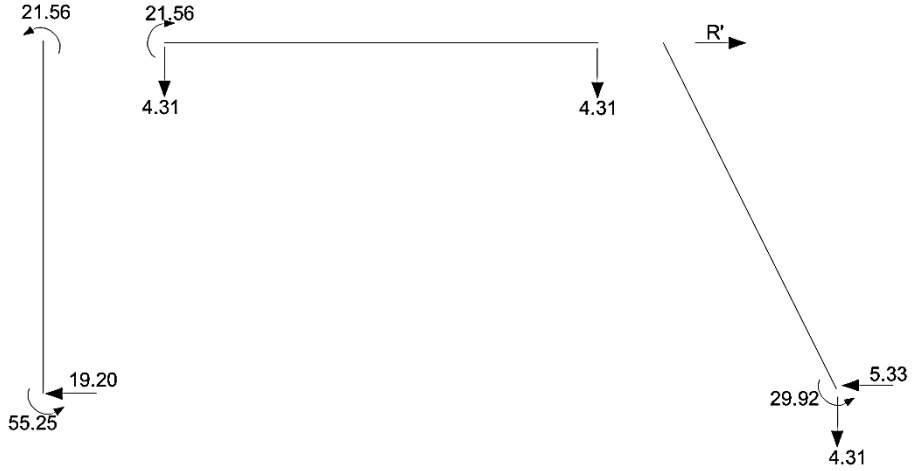
-momenti na krajevima štapova

$M_{AB}'=M_{BA}'=-6*k_{AB}*\Psi_{AB}=-6*0.593*(-25)=88.95$ kNm
$M_{BC}'=-3*k_{BC}*\Psi_{BC}=-3*0.15*10=-4.5$ kNm
$M_{DC}'=-3*k_{DC}*\Psi_{DC}=-3*0.398*(-25.06)=29.92$ kNm

M_1=88.95-4.5=84.45 kNm => iteracijski moment M=-84.45 kNm



-određivanje sile R'



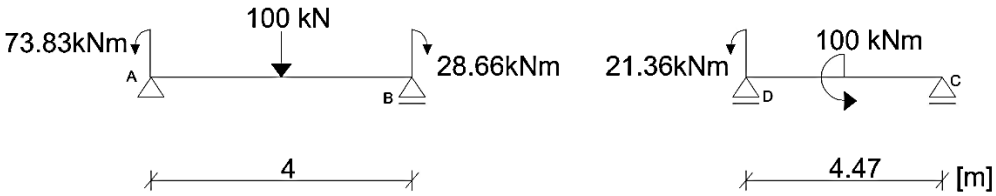
$\sum F_X=0$
R'=19.20+5.33=24.53 kN

7.) Izračun stvarnog pomaka u i momenata na sustavu

R+R'u=0
-7.26+24.53u=0
u=0.296

M_{AB}=57.48+(55.25*0.296)=73.33 kNm
M_{BA}=-35.04+(21.56*0.296)=-28.66 kNm
M_{BC}=35.04+(-21.56*0.296)=28.66 kNm
M_{DC}=12.5+(29.92*0.296)=21.36 kNm

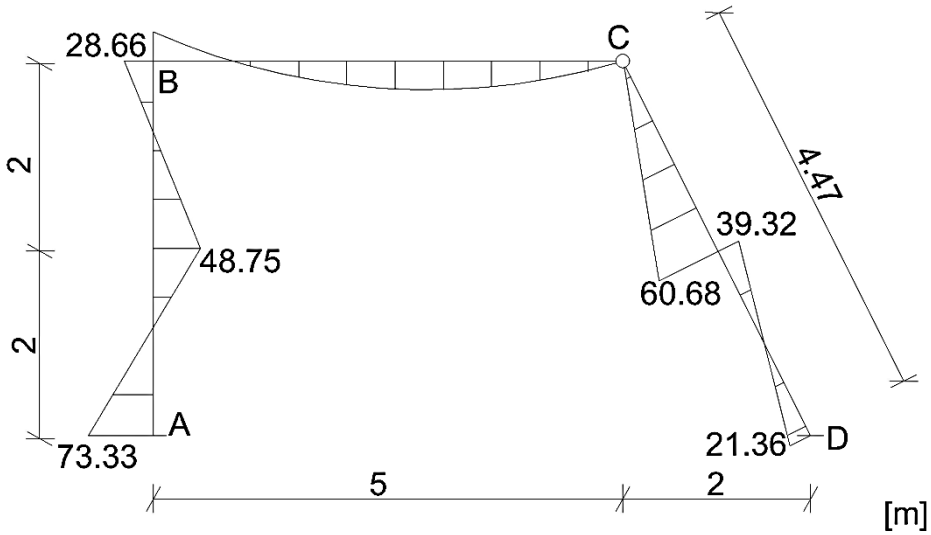
-izračun momenata na mjestima opterećenja



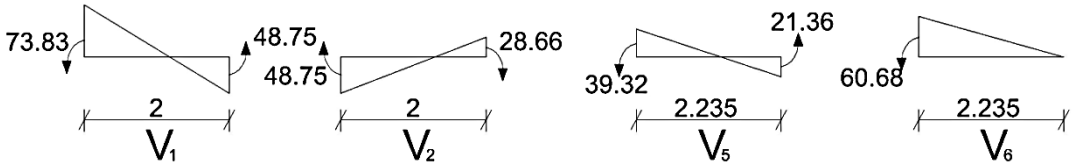
$\sum M_B=0$
$-V_A*4+100*2-28.66+73.83=0$
$V_A=61.29$ kN
$M_{AB/2}=V_A*2-73.83=48.75$ kNm

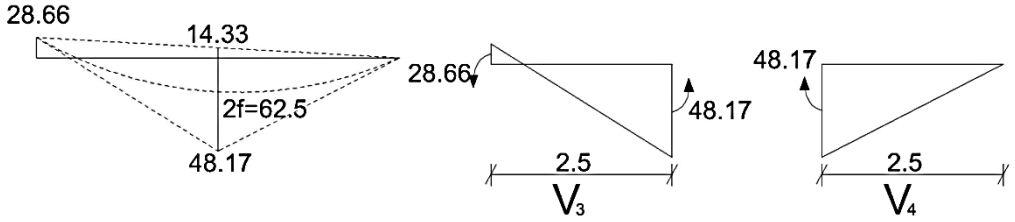
$\sum M_C=0$
$-V_D*4.47+100+21.36=0$
$V_D=27.15$ kN
$M_{BC}^L=V_D*2.235-21.36=39.32$ kNm
$M_{BC}^L=-39.32+100=60.68$ kNm

- momentni dijagram bez ekstremnog momenta



8.) Difernecijalni M-V odnosi



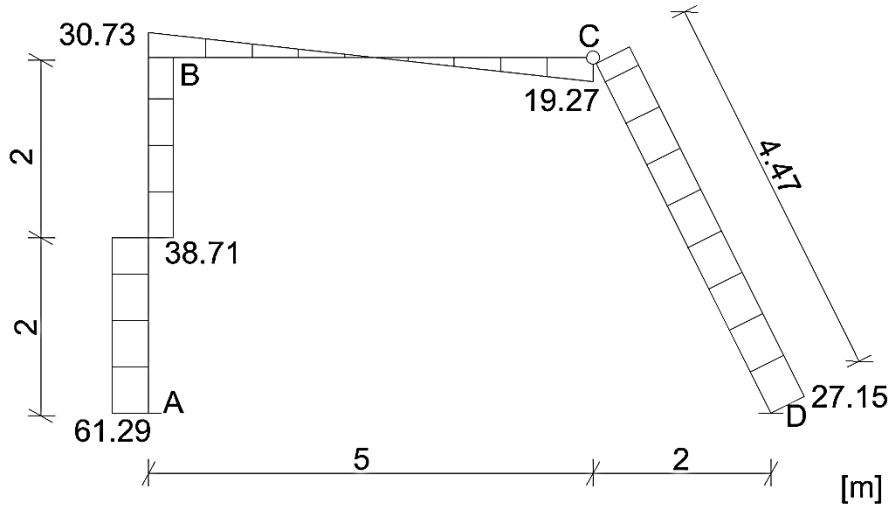


$V_1=\frac{73.83+48.75}{2}$=61.29 kN $V_2=\frac{-48.75-28.66}{2}$=-38.71 kN

$V_3=\frac{28.66+48.17}{2.5}$=30.73 kN $V_4=\frac{-48.17}{2.5}$=-19.27 kN

$V_5=\frac{39.32+21.36}{2.235}$=27.15 kN $V_6=\frac{60.68}{2.235}$=27.15 kN

-konačni dijagram poprečnih sila

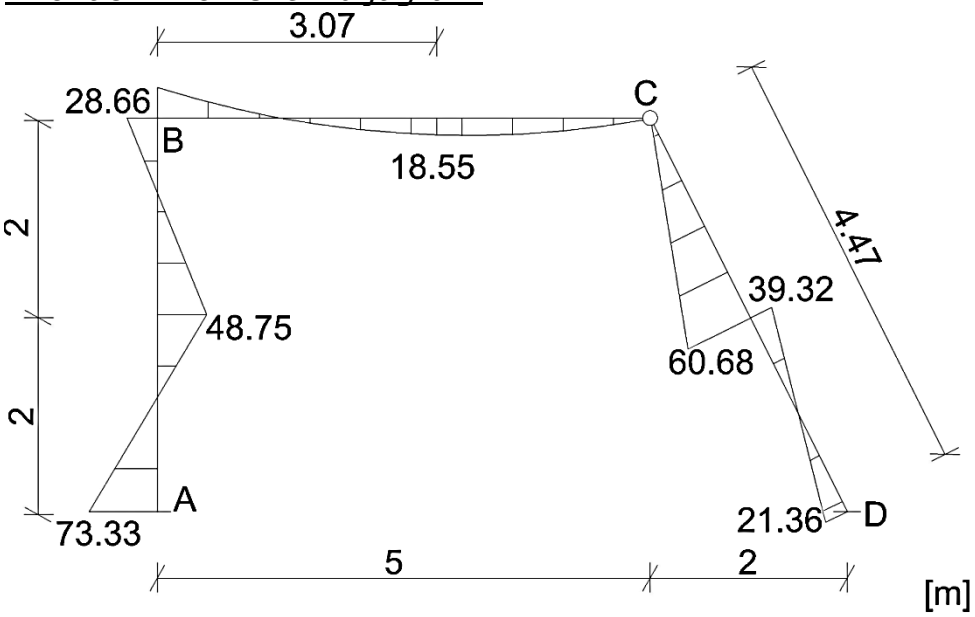


-mjesto ekstremnog momenta M_x

$\frac{50}{5}=\frac{30}{x}$ => $x=3.07m$

M_x=61.29*4+30.73*3.07-73.83-100*2-10*3.07*1.535=18.55 kNm

- konačni momentni dijagram



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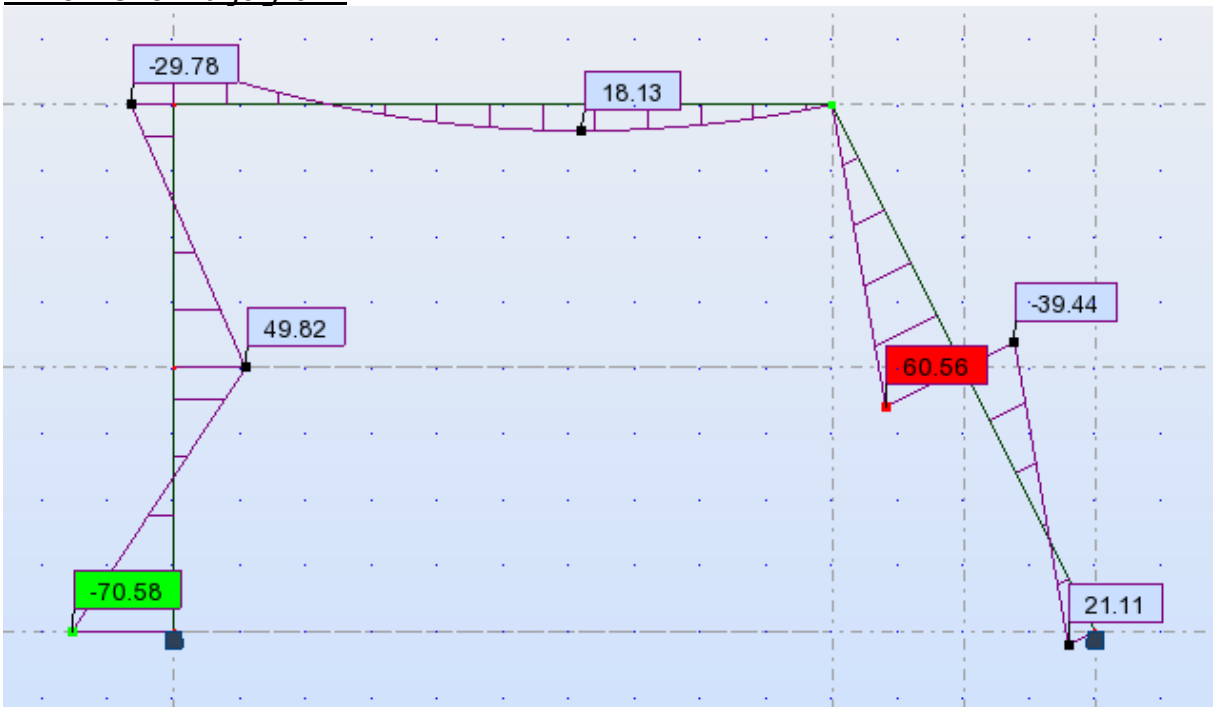
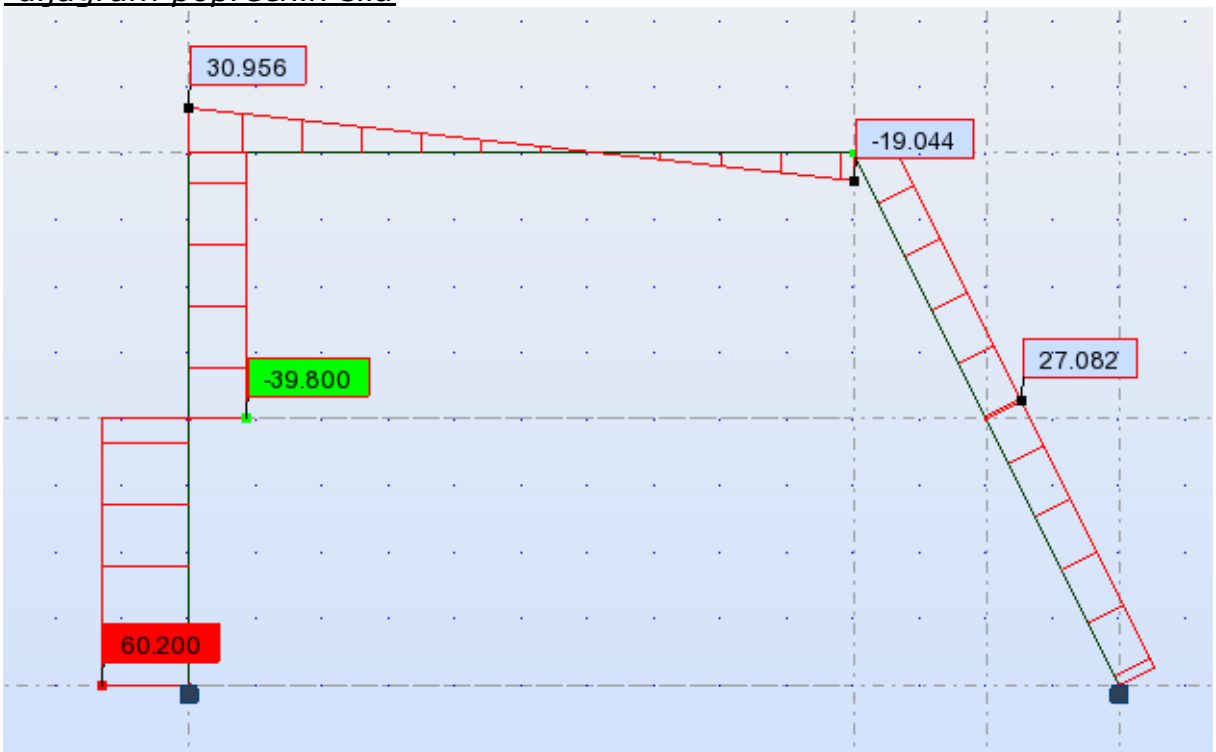
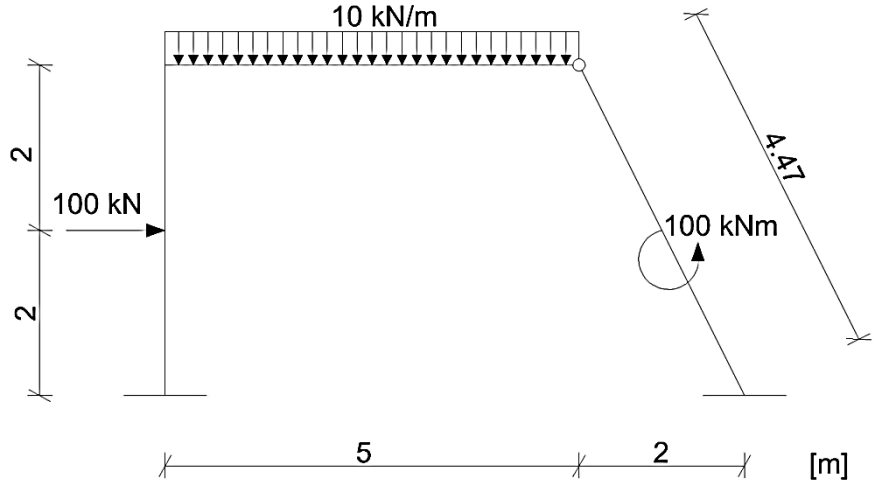
- momentni dijagram



-dijagram poprečnih sila



USPOREDBA REZULTATA PRORAČUNA STATIČKI NEODREĐENOG SUSTAVA DOBIVENIM RAZLIČITIM METODAMA



	Metoda sila	Metoda pomaka	Cross-ova metoda	ROBOT	
M_A	-70.33	-70.21	-73.33	-70.58	kNm
$M_{AB/2}$	50.0	49.97	48.75	49.82	kNm
M_B	-29.70	-29.86	-28.66	-29.78	kNm
M_X	18.34	18.14	18.55	18.13	kNm
M_D	21.43	21.56	21.36	21.11	kNm
$M_{DC/2}^{DOLJE}$	-39.29	-39.19	-39.34	-39.44	kNm
$M_{DC/2}^{GORE}$	60.71	60.81	60.68	60.56	kNm

ZAKLJUČAK:

Uzimajući u obzir računalni program Robot kao referentno rješenje, može se zaključiti da „Metoda sila", „Metoda pomaka" i „Cross-ova metoda" daju odgovarajuće rezultate. Odstupanja u odnosu na rješenja iz Robota za „Metodu sila" i „Metodu pomaka" su bazirana iza decimalne točke, te samim time zanemariva. „Cross-ova metoda" u nekim točkama ima odstupanja manja od 5%, što je dopušteno odstupanje.
Sve tri metode su uspješno izračunale rezne sile statički neodređenog sustava.

LITERATURA

- Lozančić S., Kalman T., Grubišić M.: Nastavni materijali http://www.gfos.unios.hr/portal/index.php/nastava/studiji/sveucilisni-preddiplomski-studij/gradjevna-statika-ii.html

- Milutin Anđelić ,Građevna statika II; Građevinski fakultet Sveučilišta u Zagrebu

- K. Fresl: GS – Bilješke i skice predavanja, http://master.grad.hr/nastava/gs